

5.4 Binomial Coefficients

As we remarked in Section 5.3, the number of r -combinations from a set with n elements is often denoted by $\binom{n}{r}$. This number is also called a **binomial coefficient** because these numbers occur as coefficients in the expansion of powers of binomial expressions such as $(a + b)^n$. We will discuss the **Binomial Theorem**, which gives a power of a binomial expression as a sum of terms involving binomial coefficients. We will prove this theorem using a combinatorial proof. We will also show how combinatorial proofs can be used to establish some of the many different identities that express relationships among binomial coefficients.

The Binomial Theorem

 **Links** The Binomial Theorem gives the coefficients of the expansion of powers of binomial expressions. A **binomial expression** is simply the sum of two terms, such as $x + y$. (The terms can be products of constants and variables, but that does not concern us here.) Example 1 illustrates why this theorem holds.

EXAMPLE 1 The expansion of $(x + y)^3$ can be found using combinatorial reasoning instead of multiplying the three terms out. When $(x + y)^3 = (x + y)(x + y)(x + y)$ is expanded, all products of a term in the first sum, a term in the second sum, and a term in the third sum are added. Terms of the form x^3 , x^2y , xy^2 , and y^3 arise. To obtain a term of the form x^3 , an x must be chosen in each of the sums, and this can be done in only one way. Thus, the x^3 term in the product has a coefficient of 1. To obtain a term of the form x^2y , an x must be chosen in two of the three sums (and consequently a y in the other sum). Hence, the number of such terms is the number of 2-combinations of three objects, namely, $\binom{3}{2}$. Similarly, the number of terms of the form xy^2 is the number of ways to pick one of the three sums to obtain an x (and consequently take a y from each of the other two sums). This can be done in $\binom{3}{1}$ ways. Finally, the only way to obtain a y^3 term is to choose the y for each of the three sums in the product, and this can be done in exactly one way. Consequently, it follows that

$$\begin{aligned}(x + y)^3 &= (x + y)(x + y)(x + y) = (xx + xy + yx + yy)(x + y) \\ &= xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy \\ &= x^3 + 3x^2y + 3xy^2 + y^3.\end{aligned}$$

The Binomial Theorem will now be stated.

THEOREM 1 THE BINOMIAL THEOREM Let x and y be variables, and let n be a nonnegative integer. Then

$$\begin{aligned}(x + y)^n &= \sum_{j=0}^n \binom{n}{j} x^{n-j} y^j \\ &= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.\end{aligned}$$

Proof: A combinatorial proof of the theorem will be given. The terms in the product when it is expanded are of the form $x^{n-j} y^j$ for $j = 0, 1, 2, \dots, n$. To count the number of terms of the form $x^{n-j} y^j$, note that to obtain such a term it is necessary to choose $n - j$ x s from the n sums (so that the other j terms in the product are y s). Therefore, the coefficient of $x^{n-j} y^j$ is $\binom{n}{n-j}$, which is equal to $\binom{n}{j}$. This proves the theorem. \triangleleft

The use of the Binomial Theorem is illustrated by Examples 2–4.

EXAMPLE 2 What is the expansion of $(x + y)^4$?

Extra Examples  *Solution:* From the Binomial Theorem it follows that

$$\begin{aligned}(x + y)^4 &= \sum_{j=0}^4 \binom{4}{j} x^{4-j} y^j \\ &= \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4 \\ &= x^4 + 4x^3 y + 6x^2 y^2 + 4x y^3 + y^4.\end{aligned}$$

EXAMPLE 3 What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x + y)^{25}$?

Solution: From the Binomial Theorem it follows that this coefficient is

$$\binom{25}{13} = \frac{25!}{13! 12!} = 5,200,300.$$

EXAMPLE 4 What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Solution: First, note that this expression equals $(2x + (-3y))^{25}$. By the Binomial Theorem, we have

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j} (-3y)^j.$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$, namely,

$$\binom{25}{13} 2^{12} (-3)^{13} = -\frac{25!}{13! 12!} 2^{12} 3^{13}.$$

We can prove some useful identities using the Binomial Theorem, as Corollaries 1, 2, and 3 demonstrate.

COROLLARY 1 Let n be a nonnegative integer. Then

$$\sum_{k=0}^n \binom{n}{k} = 2^n.$$

Proof: Using the Binomial Theorem with $x = 1$ and $y = 1$, we see that

$$2^n = (1 + 1)^n = \sum_{k=0}^n \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^n \binom{n}{k}.$$

This is the desired result.

There is also a nice combinatorial proof of Corollary 1, which we now present.

Proof: A set with n elements has a total of 2^n different subsets. Each subset has zero elements, one element, two elements, ..., or n elements in it. There are $\binom{n}{0}$ subsets with zero elements, $\binom{n}{1}$ subsets with one element, $\binom{n}{2}$ subsets with two elements, ..., and $\binom{n}{n}$ subsets with n elements. Therefore,

$$\sum_{k=0}^n \binom{n}{k}$$

counts the total number of subsets of a set with n elements. This shows that

$$\sum_{k=0}^n \binom{n}{k} = 2^n.$$

COROLLARY 2 Let n be a positive integer. Then

$$\sum_{k=0}^n (-1)^k \binom{n}{k} = 0.$$

Proof: When we use the Binomial Theorem with $x = -1$ and $y = 1$, we see that

$$0 = 0^n = ((-1) + 1)^n = \sum_{k=0}^n \binom{n}{k} (-1)^k 1^{n-k} = \sum_{k=0}^n \binom{n}{k} (-1)^k.$$

This proves the corollary.

Remark: Corollary 2 implies that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots.$$

COROLLARY 3 Let n be a nonnegative integer. Then

$$\sum_{k=0}^n 2^k \binom{n}{k} = 3^n.$$

Proof: We recognize that the left-hand side of this formula is the expansion of $(1 + 2)^n$ provided by the Binomial Theorem. Therefore, by the Binomial Theorem, we see that

$$(1 + 2)^n = \sum_{k=0}^n \binom{n}{k} 1^{n-k} 2^k = \sum_{k=0}^n \binom{n}{k} 2^k.$$

Hence

$$\sum_{k=0}^n 2^k \binom{n}{k} = 3^n.$$

Pascal's Identity and Triangle

The binomial coefficients satisfy many different identities. We introduce one of the most important of these now.

THEOREM 2 Let n and k be positive integers with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Proof: Suppose that T is a set containing $n+1$ elements. Let a be an element in T , and let $S = T - \{a\}$. Note that there are $\binom{n+1}{k}$ subsets of T containing k elements. However, a subset of T with k elements either contains a together with $k-1$ elements of S , or contains k elements of S and does not contain a . Because there are $\binom{n}{k-1}$ subsets of $k-1$ elements of S , there are $\binom{n}{k-1}$ subsets of k elements of T that contain a . And there are $\binom{n}{k}$ subsets of k elements of T that do not contain a , because there are $\binom{n}{k}$ subsets of k elements of S . Consequently,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Remark: A combinatorial proof of Pascal's Identity has been given. It is also possible to prove this identity by algebraic manipulation from the formula for $\binom{n}{r}$ (see Exercise 19 at the end of this section).

Remark: Pascal's Identity, together with the initial conditions $\binom{n}{0} = \binom{n}{n} = 1$ for all integers n , can be used to recursively define binomial coefficients. This recursive definition is useful in the computation of binomial coefficients because only addition, and not multiplication, of integers is needed to use this recursive definition.

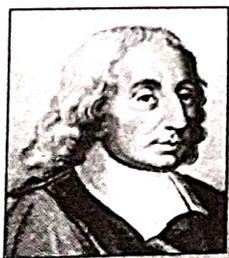
Pascal's Identity is the basis for a geometric arrangement of the binomial coefficients in a triangle, as shown in Figure 1.

The n th row in the triangle consists of the binomial coefficients

$$\binom{n}{k}, \quad k = 0, 1, \dots, n.$$

This triangle is known as **Pascal's triangle**. Pascal's Identity shows that when two adjacent binomial coefficients in this triangle are added, the binomial coefficient in the next row between these two coefficients is produced.

links



BLAISE PASCAL (1623–1662) Blaise Pascal exhibited his talents at an early age, although his father, who had made discoveries in analytic geometry, kept mathematics books away from him to encourage other interests. At 16 Pascal discovered an important result concerning conic sections. At 18 he designed a calculating machine, which he built and sold. Pascal, along with Fermat, laid the foundations for the modern theory of probability. In this work he made new discoveries concerning what is now called Pascal's triangle. In 1654, Pascal abandoned his mathematical pursuits to devote himself to theology. After this, he returned to mathematics only once. One night, distracted by a severe toothache, he sought comfort by studying the mathematical properties of the cycloid. Miraculously, his pain subsided, which he took as a sign of divine approval of the study of mathematics.

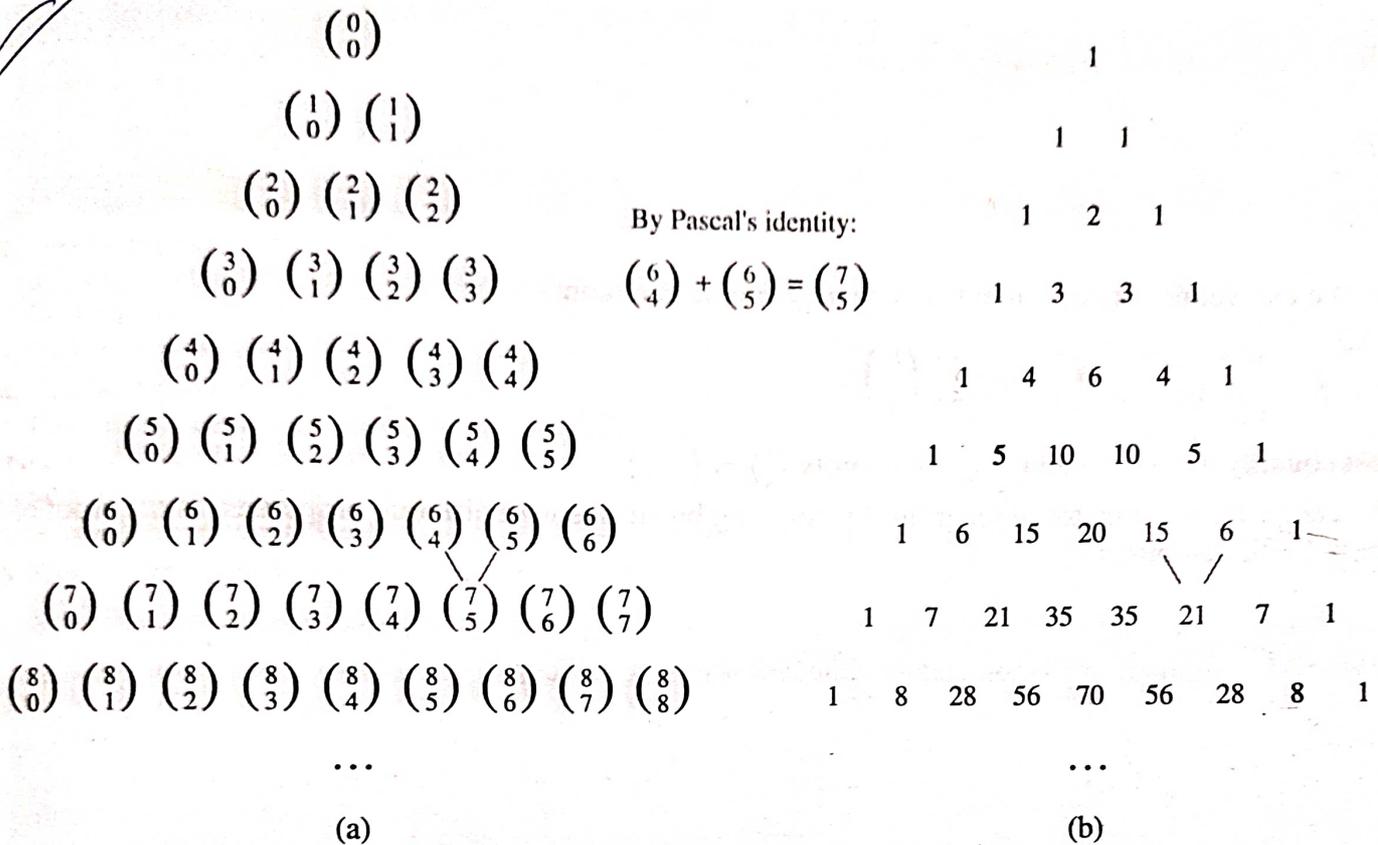


FIGURE 1 Pascal's Triangle.

Some Other Identities of the Binomial Coefficients

We conclude this section with combinatorial proofs of two of the many identities enjoyed by the binomial coefficients.

THEOREM 3 VANDERMONDE'S IDENTITY Let $m, n,$ and r be nonnegative integers with r not exceeding either m or n . Then

$$\binom{m+n}{r} = \sum_{k=0}^r \binom{m}{r-k} \binom{n}{k}.$$

Remark: This identity was discovered by mathematician Alexandre-Théophile Vandermonde in the eighteenth century.

Proof: Suppose that there are m items in one set and n items in a second set. Then the total number of ways to pick r elements from the union of these sets is $\binom{m+n}{r}$. Another way to pick r elements from the union is to pick k elements from the first set and then $r - k$ elements from the second set, where k is an integer with $0 \leq k \leq r$. This can be done in $\binom{m}{k} \binom{n}{r-k}$ ways, using the product rule. Hence, the total number of ways to pick r elements from the union also equals

$$\binom{m+n}{r} = \sum_{k=0}^r \binom{m}{r-k} \binom{n}{k}.$$

This proves Vandermonde's Identity.

Corollary 4 follows from Vandermonde's Identity. ◀

COROLLARY 4 If n is a nonnegative integer, then

$$\binom{2n}{n} = \sum_{k=0}^n \binom{n}{k}^2$$

Proof: We use Vandermonde's Identity with $m = r = n$ to obtain

$$\binom{2n}{n} = \sum_{k=0}^n \binom{n}{n-k} \binom{n}{k} = \sum_{k=0}^n \binom{n}{k}^2.$$

The last equality was obtained using the identity $\binom{n}{k} = \binom{n}{n-k}$. \triangleleft

We can prove combinatorial identities by counting bit strings with different properties, as the proof of Theorem 4 will demonstrate.

THEOREM 4 Let n and r be nonnegative integers with $r \leq n$. Then

$$\binom{n+1}{r+1} = \sum_{j=r}^n \binom{j}{r}.$$

Proof: We use a combinatorial proof. By Example 14 in Section 5.3, the left-hand side, $\binom{n+1}{r+1}$, counts the bit strings of length $n+1$ containing $r+1$ ones.

We show that the right-hand side counts the same objects by considering the cases corresponding to the possible locations of the final 1 in a string with $r+1$ ones. This final one must occur at position $r+1$, $r+2$, ..., or $n+1$. Furthermore, if the last one is the k th bit there must be r ones among the first $k-1$ positions. Consequently, by Example 14 in Section 5.3, there are $\binom{k-1}{r}$ such bit strings. Summing over k with $r+1 \leq k \leq n+1$, we find that there are

$$\sum_{k=r+1}^{n+1} \binom{k-1}{r} = \sum_{j=r}^n \binom{j}{r}$$

bit strings of length n containing exactly $r+1$ ones. (Note that the last step follows from the change of variables $j = k-1$.) Because the left-hand side and the right-hand side count the same objects, they are equal. This completes the proof. \triangleleft



ALEXANDRE-THÉOPHILE VANDERMONDE (1735–1796) Because Alexandre-Théophile Vandermonde was a sickly child, his physician father directed him to a career in music. However, he later developed an interest in mathematics. His complete mathematical work consists of four papers published in 1771–1772. These papers include fundamental contributions on the roots of equations, on the theory of determinants, and on the knight's tour problem (introduced in the exercises in Section 9.5). Vandermonde's interest in mathematics lasted for only 2 years. Afterward, he published papers on harmony, experiments with cold, and the manufacture of steel. He also became interested in politics, joining the cause of the French revolution and holding several different positions in government.

Exercises

- Find the expansion of $(x + y)^4$
 - using combinatorial reasoning, as in Example 1.
 - using the Binomial Theorem.
- Find the expansion of $(x + y)^5$
 - using combinatorial reasoning, as in Example 1.
 - using the Binomial Theorem.
- Find the expansion of $(x + y)^6$.
- Find the coefficient of x^5y^8 in $(x + y)^{13}$.
- How many terms are there in the expansion of $(x + y)^{100}$ after like terms are collected?
- What is the coefficient of x^7 in $(1 + x)^{11}$?
- What is the coefficient of x^9 in $(2 - x)^{19}$?
- What is the coefficient of x^8y^9 in the expansion of $(3x + 2y)^{17}$?
- What is the coefficient of $x^{101}y^{99}$ in the expansion of $(2x - 3y)^{200}$?
- *10. Give a formula for the coefficient of x^k in the expansion of $(x + 1/x)^{100}$, where k is an integer.
- *11. Give a formula for the coefficient of x^k in the expansion of $(x^2 - 1/x)^{100}$, where k is an integer.
12. The row of Pascal's triangle containing the binomial coefficients $\binom{10}{k}$, $0 \leq k \leq 10$, is:
1 10 45 120 210 252 210 120 45 10 1
Use Pascal's Identity to produce the row immediately following this row in Pascal's triangle.
13. What is the row of Pascal's triangle containing the binomial coefficients $\binom{9}{k}$, $0 \leq k \leq 9$?
14. Show that if n is a positive integer, then $1 = \binom{n}{0} < \binom{n}{1} < \dots < \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lfloor n/2 \rfloor} > \dots > \binom{n}{n-1} > \binom{n}{n} = 1$.
15. Show that $\binom{n}{k} \leq 2^n$ for all positive integers n and all integers k with $0 \leq k \leq n$.
16. a) Use Exercise 14 and Corollary 1 to show that if n is an integer greater than 1, then $\binom{n}{\lfloor n/2 \rfloor} \geq 2^n/n$.
b) Conclude from part (a) that if n is a positive integer, then $\binom{2n}{n} \geq 4^n/2n$.
- *17. Show that if n and k are integers with $1 \leq k \leq n$, then $\binom{n}{k} \leq n^k/2^{k-1}$.
18. Suppose that b is an integer with $b \geq 7$. Use the Binomial Theorem and the appropriate row of Pascal's triangle to find the base- b expansion of $(11)_b^4$ [that is, the fourth power of the number $(11)_b$ in base- b notation].
19. Prove Pascal's Identity, using the formula for $\binom{n}{r}$.
20. Suppose that k and n are integers with $1 \leq k < n$. Prove the hexagon identity

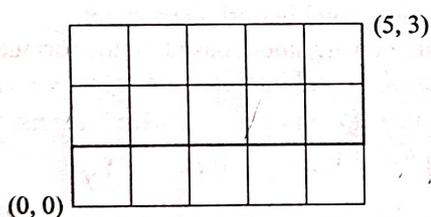
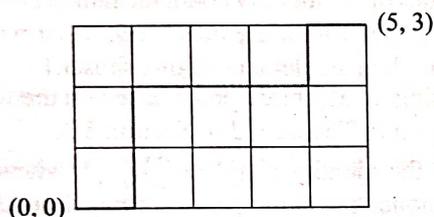
$$\binom{n-1}{k-1} \binom{n}{k+1} \binom{n+1}{k} = \binom{n-1}{k} \binom{n}{k-1} \binom{n+1}{k+1}$$
21. Prove that if n and k are integers with $1 \leq k \leq n$, then $k \binom{n}{k} = n \binom{n-1}{k-1}$
 - using a combinatorial proof. [Hint: Show that the two sides of the identity count the number of ways to select a subset with k elements from a set with n elements and then an element of this subset.]
 - using an algebraic proof based on the formula for $\binom{n}{r}$ given in Theorem 2 in Section 5.3.
22. Prove the identity $\binom{n}{r} \binom{r}{k} = \binom{n}{k} \binom{n-k}{r-k}$, whenever n , r , and k are nonnegative integers with $r \leq n$ and $k \leq r$,
 - using a combinatorial argument.
 - using an argument based on the formula for the number of r -combinations of a set with n elements.
23. Show that if n and k are positive integers, then

$$\binom{n+1}{k} = (n+1) \binom{n}{k-1} / k$$
Use this identity to construct an inductive definition of the binomial coefficients.
24. Show that if p is a prime and k is an integer such that $1 \leq k \leq p-1$, then p divides $\binom{p}{k}$.
25. Let n be a positive integer. Show that $\binom{2n}{n+1} + \binom{2n}{n} = \binom{2n+2}{n+1}/2$.
- *26. Let n and k be integers with $1 \leq k \leq n$. Show that $\sum_{k=1}^n \binom{n}{k} \binom{n}{k-1} = \binom{2n+2}{n+1}/2 - \binom{2n}{n}$.
- *27. Prove that

$$\sum_{k=0}^r \binom{n+k}{k} = \binom{n+r+1}{r}$$
whenever n and r are positive integers,
 - using a combinatorial argument.
 - using Pascal's identity.
28. Show that if n is a positive integer, then $\binom{2n}{2} = 2 \binom{n}{2} + n^2$
 - using a combinatorial argument.
 - by algebraic manipulation.
- *29. Give a combinatorial proof that $\sum_{k=1}^n k \binom{n}{k} = n2^{n-1}$. [Hint: Count in two ways the number of ways to select a committee and to then select a leader of the committee.]
- *30. Give a combinatorial proof that $\sum_{k=1}^n k \binom{n}{k}^2 = n \binom{2n-1}{n-1}$. [Hint: Count in two ways the number of ways to select a committee, with n members from a group of n mathematics professors and n computer science professors, such that the chairperson of the committee is a mathematics professor.]
31. Show that a nonempty set has the same number of subsets with an odd number of elements as it does subsets with an even number of elements.

*32. Prove the Binomial Theorem using mathematical induction.

33. In this exercise we will count the number of paths in the xy plane between the origin $(0, 0)$ and point (m, n) such that each path is made up of a series of steps, where each step is a move one unit to the right or a move one unit upward. (No moves to the left or downward are allowed.) Two such paths from $(0, 0)$ to $(5, 3)$ are illustrated here.



- a) Show that each path of the type described can be represented by a bit string consisting of m 0s and n 1s, where a 0 represents a move one unit to the right and a 1 represents a move one unit upward.
- b) Conclude from part (a) that there are $\binom{m+n}{n}$ paths of the desired type.
34. Use Exercise 33 to prove that $\binom{n}{k} = \binom{n}{n-k}$ whenever k is an integer with $0 \leq k \leq n$. [Hint: Consider the number of paths of the type described in Exercise 33 from $(0, 0)$ to $(n-k, k)$ and from $(0, 0)$ to $(k, n-k)$.]

35. Use Exercise 33 to prove Theorem 4. [Hint: Count the number of paths with n steps of the type described in Exercise 33. Every such path must end at one of the points $(n-k, k)$ for $k = 0, 1, 2, \dots, n$.]
36. Use Exercise 33 to prove Pascal's Identity. [Hint: Show that a path of the type described in Exercise 33 from $(0, 0)$ to $(n+1-k, k)$ passes through either $(n+1-k, k-1)$ or $(n-k, k)$, but not through both.]
37. Prove the identity in Exercise 27 using Exercise 33. [Hint: First, note that the number of paths from $(0, 0)$ to $(n+1, r)$ equals $\binom{n+1+r}{r}$. Second, count the number of paths by summing the number of these paths that start by going k units upward for $k = 0, 1, 2, \dots, r$.]
38. Give a combinatorial proof that if n is a positive integer then $\sum_{k=0}^n k^2 \binom{n}{k} = n(n+1)2^{n-2}$. [Hint: Show that both sides count the ways to select a subset of a set of n elements together with two not necessarily distinct elements from this subset. Furthermore, express the right-hand side as $n(n-1)2^{n-2} + n2^{n-1}$.]
- *39. Determine a formula involving binomial coefficients for the n th term of a sequence if its initial terms are those listed. [Hint: Looking at Pascal's triangle will be helpful. Although infinitely many sequences start with a specified set of terms, each of the following lists is the start of a sequence of the type desired.]
- a) 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, ...
- b) 1, 4, 10, 20, 35, 56, 84, 120, 165, 220, ...
- c) 1, 2, 6, 20, 70, 252, 924, 3432, 12870, 48620, ...
- d) 1, 1, 2, 3, 6, 10, 20, 35, 70, 126, ...
- e) 1, 1, 1, 3, 1, 5, 15, 35, 1, 9, ...
- f) 1, 3, 15, 84, 495, 3003, 18564, 116280, 735471, 4686825, ...