* Euler's Theorem:

A function homogenous of some degree has a property sometimes used in economics which was first develped by Leonhard Euler.

The Theorem:

A differentiable function f of n variables is homogenous of degree λ, if and only if (iff):

$$\sum_{i=1}^{n} x_i f_i'(x_1, x_2, \ldots, x_n) = \lambda f(x_1, x_2, \ldots, x_n) \quad \forall\ x_1, x_2, \ldots, x_n$$

Proof:

Let f be homogenous of degree λ.

$$\Rightarrow f(tx_1, tx_2, \ldots, tx_n) = t^{\lambda} f(x_1, x_2, \ldots, x_n) \quad \forall\ t>0$$

Differentiating both side of the eqn w.r.t. t:

$$\frac{\partial f(tx_1, tx_2, \ldots, tx_n)}{\partial t} = \frac{\partial \{t^{\lambda} f(x_1, x_2, \ldots, x_n)\}}{\partial t}.$$

$$\Rightarrow x_1 f_1'(tx_1, tx_2, \ldots, tx_n) + x_2 f_2'(tx_1, tx_2, \ldots, tx_n) + \cdots$$
$$\cdots + x_n f_n'(tx_1, tx_2, \ldots, tx_n) = \lambda t^{\lambda-1} f(x_1, \ldots, x_n).$$

Now, setting $t=1$:

$$x_1 f_1'(x_1, x_2, \ldots, x_n) + x_2 f_2'(x_1, x_2, \ldots, x_n) + \cdots + x_n f_n'(x_1, x_2, \ldots x_n)$$
$$= \lambda f(x_1, x_2, \ldots, x_n).$$

$$\Rightarrow \sum_{i=1}^{n} x_i f_i(x_1, x_2, \ldots, x_n) = \lambda f(x_1, x_2, \ldots, x_n) \quad //$$

Proved

In Economics Euler's theorem is used generally for linear homogenous production functions:

If $Q = f(K, L)$ be a firm's production fn. which is homogenous of degree 1 (linear homogeneous), according to Euler's theorem:

$$K \frac{\partial Q}{\partial K} + L \frac{\partial Q}{\partial L} = Q.$$

$$Q = f(K, L).$$

Proof:

$$\frac{Q}{L} = f\left(\frac{K}{L}, \frac{L}{L}\right).$$

[∵ Q is homo. of degree 1]

$\frac{\partial Q}{\partial K}$

$$\Rightarrow Q = L \cdot g(K/L).$$

$$\frac{\partial Q}{\partial L} = \frac{\partial}{\partial L}\left[L \cdot g(K/L)\right] = g(K/L) + L \cdot g'(K/L)\left(\frac{\partial K/L}{\partial L}\right)$$

$$= g(K/L) + L \cdot g'(K/L)\left(-\frac{K}{L^2}\right).$$

$$= g(K/L) - \frac{K}{L} g'(K/L).$$

$$= F(K/L) \quad \text{—— (i)}$$

$\Rightarrow \frac{\partial Q}{\partial L}$ (MP_L) is fn. of $\left(\frac{K}{L}\right)$ alone.

~~$\frac{Q}{K} = \frac{f(K,L)}{K}$~~

~~$\frac{Q}{L} = \frac{f(K,L)}{K}$.~~

~~Now, if Q is linearly homogenous: multiplying K & L by $\frac{1}{K}$, we get: f(K,L), we get:~~

~~$f\left(K \cdot \frac{1}{K}, L \cdot \frac{1}{K}\right) = \frac{1}{K} f(K, L)$.~~

[Contd. in next page]

Note: If $Q = f(K, L)$ is linearly homogenous, then:

$$f\left(\frac{K}{L}, \frac{L}{L}\right) = \frac{1}{L} Q.$$

[multiplying K & L by $\frac{1}{L}$ the linearly homogenous fn. itself is multiplied by $\frac{1}{L}$]. $\lambda = 1$ here

$$\Rightarrow f\left(\frac{K}{L}, 1\right) = \frac{Q}{L}.$$

$$\Rightarrow \frac{Q}{L} = g(k) \quad \left[k = \frac{K}{L}\right].$$

Similarly, we can show $\frac{Q}{K}$ is also a fn. of $\frac{K}{L}$.

Again. $\frac{Q}{L} = g(K/L) \Rightarrow Q = L \cdot g(K/L)$.

$\frac{\partial Q}{\partial K} = \frac{\partial}{\partial K}\left[L \cdot g(K/L)\right]$.

$= L \cdot g'(K/L) \cdot \frac{\partial (K/L)}{\partial L}$

$= L \cdot g'(K/L) \cdot \frac{1}{L} = g'(K/L)$. — (ii).

$\Rightarrow \frac{\partial Q}{\partial K}$ (or MP_K) is a fn. $g\left(\frac{K}{L}\right)$ alone.

Now, we prove Euler's Theorem : [for linear homogeneous prodn fn.]

To show that $K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L} = Q$.

We use eqn. (ii) & (iii) here ~~and~~ by replacing $\frac{\partial Q}{\partial K}$ & $\frac{\partial Q}{\partial L}$ in ~~K~~

$\therefore K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L} = K \cdot g'(K/L) + L \cdot \left[g(K/L) - \frac{K}{L} g'(K/L)\right]$

$= \cancel{K g'(K/L)} + L g(K/L) - \cancel{K g'(K/L)}$

$= L \cdot g(K/L) = Q$. $\therefore$ Proved.

Special cases of homogeneous Prodn Fn:

Cobb-Douglas Prodn. Fn:

It takes the form: $Q = AK^{\alpha}L^{\beta}$, where A is a positive constant and α & β are positive fractions

Note: $\alpha + \beta = 1 \Rightarrow$ CRS.
$\alpha + \beta > 1 \Rightarrow$ IRS.
$\alpha + \beta < 1 \Rightarrow$ DRS.

for linear homogeneous CD prodn. fn. $Q = AK^{\alpha}L^{1-\alpha}$.

We test $\frac{Q}{K}$:
$$\frac{Q}{K} = \frac{A \cdot K^{\alpha}L^{1-\alpha}}{K} = AK^{\alpha-1}L^{1-\alpha}$$
$$= \frac{AK^{\alpha-1}}{L^{\alpha-1}} = A(K/L)^{\alpha-1}$$
$$= \psi(K/L) \quad \because A \text{ is constant.}$$

Here: $\boxed{\frac{Q}{K} = A\left(\frac{K}{L}\right)^{\alpha-1}}$

Again, $$\frac{Q}{L} = \frac{AK^{\alpha}L^{1-\alpha}}{L} = \frac{AK^{\alpha}}{L^{1-1+\alpha}} = A\left(\frac{K}{L}\right)^{\alpha}$$
$$= h(K/L) \quad \because A \text{ is constant.}$$

Here, $\boxed{\frac{Q}{L} = A\left(\frac{K}{L}\right)^{\alpha}}$

Now, $Q = AK^{\alpha}L^{1-\alpha}$
$$\Rightarrow \frac{\partial Q}{\partial K} = A\alpha K^{\alpha-1}L^{1-\alpha} = A\alpha\left(\frac{K}{L}\right)^{\alpha-1}$$

And, $$\frac{\partial Q}{\partial L} = A(1-\alpha)K^{\alpha}L^{-\alpha} = A(1-\alpha)\left(\frac{K}{L}\right)^{\alpha}$$

Now, again, $$K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L} = K \cdot A\alpha\left(\frac{K}{L}\right)^{\alpha-1} + L \cdot A(1-\alpha)\left(\frac{K}{L}\right)^{\alpha}$$
$$= A\alpha K^{\alpha}L^{1-\alpha} + A(1-\alpha) \cdot K^{\alpha}L^{1-\alpha}$$
$$= AK^{\alpha}L^{1-\alpha}(\cancel{\alpha} + 1 - \cancel{\alpha})$$
$$= AK^{\alpha}L^{1-\alpha} = Q.$$

$\therefore$ Proved the Euler's theorem again for C-D prodn fn.

Implicit Functions:

A fn of $y = f(x_1, x_2, \ldots, x_n)$ is called an explicit fn. Here y is explicitly expressed as fn of x.

Instead, relationship between y and $x_1, x_2, \ldots, x_n$ is given in the form $F(y, x_1, x_2, \ldots, x_n) = 0$.

Such a fn is called an implicit fn. [You can take it like : $y = f(x_1, x_2, \ldots x_n)$

$\Rightarrow y - f(x_1, x_2, \ldots, x_n) = 0.$

$\Rightarrow F(y, x_1, x_n, \ldots, x_n) = 0.$

Implicit Function Theorem:

Given $F(y, x_1, x_2, \ldots, x_n) = 0$, if:

(a) the fn F has continuous partial derivatives $F_y, F_{x_1}, F_{x_2}, \ldots, F_{x_n}$, and if:

(b) at point $(y_0, x_{10}, x_{20}, \ldots, x_{n0})$ satisfying $F(y_0, x_{10}, x_{20}, \ldots, x_{n0}) = 0$,

F_y is nonzero, Then there exists an n-dimensional neighbourhood of $(x_{10}, x_{20}, \ldots, x_{n0})$, N in which y is an implicitly defined fn of variables $x_1, x_2, \ldots x_n$, in the form of $y = f(x_1, \ldots, x_n)$ and $F(y, x_1, x_2, \ldots, x_n) = 0$ for all points in N.

Derivatives of Implicit Functions:

for, $F(y, x_1, x_2, \ldots, x_n) = 0$.

By total differentiation of F:

$dF = 0$.

or, $F_y dy + F_1 dx_1 + F_2 dx_2 + \ldots + F_n dx_n = 0$. —(i).

Now, if only y and x_1 are allowed to vary, Then eqⁿ. (i) stands : $F_y dy + F_1 dx_1 = 0$.

$$\Rightarrow \left.\frac{dy}{dx_1}\right|_{\text{other variables const.}} = \frac{\partial y}{\partial x_1} = -\frac{F_1}{F_y}.$$

In general it stands $\frac{\partial y}{\partial x_i} = -\frac{F_i}{F_y}$. $(i = 1, 2, \dots, n)$.

Example: (1). $F(y, x, w) = y^3x^2 + w^3 + yxw - 3 = 0$.

$$\frac{\partial y}{\partial x} = -\frac{F_x}{F_y} = -\frac{y^3 2x + yw}{3y^2x^2 + xw} = -\frac{2xy^3 + yw}{3y^2x^2 + xw}.$$

[Note : $F_x = \partial f/\partial x$ $F_y = \partial f / \partial y$].

Thus at point $(1, 1, 1)$: $\frac{\partial y}{\partial x} = -\frac{2+1}{3+1} = -\frac{3}{4}$.

Example : (2) Assuming the production fⁿ in implicit form: $F(Q, K, L) = 0$. find MP_K & MP_L.

$$MP_K = \frac{\partial Q}{\partial K} = -\frac{F_K}{F_Q} \quad \& \quad MP_L = \frac{\partial Q}{\partial L} = -\frac{F_L}{F_Q}.$$

Also, $MRTS_{LK} = \frac{\partial K}{\partial L} = -\frac{F_L}{F_K}$ | → slope of isoquant, (level curve).

Extending the Implicit fn. to Simultaneous equation system:

Let us consider a set of simultaneous eqns:

$$\left.\begin{aligned} &F^1(y_1, y_2, \dots, y_n;\ x_1, x_2, \dots, x_m) = 0. \\ &F^2(y_1, y_2, \dots, y_n;\ x_1, x_2, \dots, x_m) = 0. \\ &\quad \vdots \\ &F^n(y_1, y_2, \dots, y_n;\ x_1, x_2, \dots, x_m) = 0 \end{aligned}\right\} \text{n eqns}$$

We assume that F^i's are differentiable.

Then, by total differentiation:

$$\left(\frac{\partial F^1}{\partial y_1}dy_1 + \frac{\partial F^1}{\partial y_2}dy_2 + \dots + \frac{\partial F^1}{\partial y_n}dy_n\right) + \left(\frac{\partial F^1}{\partial x_1}dx_1 + \frac{\partial F^1}{\partial x_2}dx_2 + \dots + \frac{\partial F^1}{\partial x_m}dx_m\right) = 0.$$

$$\left(\frac{\partial F^2}{\partial y_1}dy_1 + \frac{\partial F^2}{\partial y_2}dy_2 + \dots + \frac{\partial F^2}{\partial y_n}dy_n\right) + \left(\frac{\partial F^2}{\partial x_1}dx_1 + \frac{\partial F^2}{\partial x_2}dx_2 + \dots + \frac{\partial F^2}{\partial x_m}dx_m\right) = 0.$$

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$$\left(\frac{\partial F^n}{\partial y_1}dy_1 + \frac{\partial F^n}{\partial y_2}dy_2 + \dots + \frac{\partial F^n}{\partial y_n}dy_n\right) + \left(\frac{\partial F^n}{\partial x_1}dx_1 + \frac{\partial F^n}{\partial x_2}dx_2 + \dots + \frac{\partial F^n}{\partial x_m}dx_m\right) = 0.$$

$$\Rightarrow \left(\frac{\partial F^1}{\partial y_1}dy_1 + \frac{\partial F^1}{\partial y_2}dy_2 + \dots + \frac{\partial F^1}{\partial y_n}dy_n\right) = -\left(\frac{\partial F^1}{\partial x_1}dx_1 + \frac{\partial F^1}{\partial x_2}dx_2 + \dots + \frac{\partial F^1}{\partial x_m}dx_m\right)$$

$$\left(\frac{\partial F^2}{\partial y_1}dy_1 + \frac{\partial F^2}{\partial y_2}dy_2 + \dots + \frac{\partial F^2}{\partial y_n}dy_n\right) = -\left(\frac{\partial F^2}{\partial x_1}dx_1 + \frac{\partial F^2}{\partial x_2}dx_2 + \dots + \frac{\partial F^2}{\partial x_m}dx_m\right)$$

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$$\left(\frac{\partial F^n}{\partial y_1}dy_1 + \frac{\partial F^n}{\partial y_2}dy_2 + \dots + \frac{\partial F^n}{\partial y_n}dy_n\right) = -\left(\frac{\partial F^n}{\partial x_1}dx_1 + \frac{\partial F^2}{\partial x_2}dx_2 + \dots + \frac{\partial F^n}{\partial x_m}dF_m\right).$$

In matrix form:

$$\begin{bmatrix} \frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_n} \\ \frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F^n}{\partial y_1} & \frac{\partial F^n}{\partial y_2} & \cdots & \frac{\partial F^n}{\partial y_n} \end{bmatrix} \begin{bmatrix} dy_1 \\ dy_2 \\ \vdots \\ dy_n \end{bmatrix} = \begin{bmatrix} \frac{\partial F^1}{\partial x_1} & \frac{\partial F^1}{\partial x_2} & \cdots & \frac{\partial F^1}{\partial x_m} \\ \frac{\partial F^2}{\partial x_1} & \frac{\partial F^2}{\partial x_2} & \cdots & \frac{\partial F^2}{\partial x_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F^n}{\partial x_1} & \frac{\partial F^n}{\partial x_2} & \cdots & \frac{\partial F^n}{\partial x_m} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_m \end{bmatrix}$$

If we want to obtain the partial derivatives w.r.t x_i, we can let $dx_t = 0$ for $t \neq i$.
Thus, we then have:

$$\begin{bmatrix} \frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_n} \\ \frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F^n}{\partial y_1} & \frac{\partial F^n}{\partial y_2} & \cdots & \frac{\partial F^n}{\partial y_n} \end{bmatrix} \begin{bmatrix} \frac{\partial y_1}{\partial x_i} \\ \frac{\partial y_2}{\partial x_i} \\ \vdots \\ \frac{\partial y_n}{\partial x_i} \end{bmatrix} = \begin{bmatrix} \frac{\partial F^1}{\partial x_i} \\ \frac{\partial F^2}{\partial x_i} \\ \vdots \\ \frac{\partial F^n}{\partial x_i} \end{bmatrix}.$$

Now, suppose the following Jacobian determinant is non-zero:

$$|J| = \left| \frac{\partial(F^1, F^2, \ldots, F^n)}{\partial(y_1, y_2, \ldots, y_n)} \right| = \begin{vmatrix} \frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_n} \\ \frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F^n}{\partial y_1} & \frac{\partial F^n}{\partial y_2} & \cdots & \frac{\partial F^n}{\partial y_n} \end{vmatrix} \neq 0.$$

Then, by Cramer's Rule:

$$\frac{\partial y_j}{\partial x_i} = \frac{|J_j^i|}{|J|} \qquad (j=1,2,\dots,n;\ i=1,2,\dots,m).$$

where $|J_j^i|$ is obtained by replacing the jth column of $|J|$ with $\left[\frac{\partial F^1}{\partial x_i}, \frac{\partial F^2}{\partial x_i}, \dots, \frac{\partial F^n}{\partial x_i}\right]'$.

We can find these derivatives by inversing the Jacobian matrix J:

$$\begin{bmatrix} \frac{\partial y_1}{\partial x_i} \\ \frac{\partial y_2}{\partial x_i} \\ \vdots \\ \frac{\partial y_n}{\partial x_i} \end{bmatrix} = - \begin{bmatrix} \frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_n} \\ \frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F^n}{\partial y_1} & \frac{\partial F^n}{\partial y_2} & \cdots & \frac{\partial F^n}{\partial y_n} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial F^1}{\partial x_i} \\ \frac{\partial F^2}{\partial x_i} \\ \vdots \\ \frac{\partial F^n}{\partial x_i} \end{bmatrix}$$

In compact notation:

$$\frac{\partial y}{\partial x_i} = -J^{-1}F_i.$$

✻ An Example following national income model:

$$Y - C - I_0 - G_0 = 0.$$
$$C - \alpha - \beta(Y-T) = 0.$$
$$T - \gamma - \delta Y = 0$$

[Note: $Y = C + I_0 + G_0$. $C = \alpha + \beta(Y-T)$. $T = \gamma + \delta Y$.]

The Jacobian determinant $|J|$:

$$|J|: \begin{vmatrix} \frac{\partial F^1}{\partial Y} & \frac{\partial F^1}{\partial C} & \frac{\partial F^1}{\partial T} \\ \frac{\partial F^2}{\partial Y} & \frac{\partial F^2}{\partial C} & \frac{\partial F^2}{\partial T} \\ \frac{\partial F^3}{\partial Y} & \frac{\partial F^3}{\partial C} & \frac{\partial F^3}{\partial T} \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ -\beta & 1 & \beta \\ -\delta & 0 & 1 \end{vmatrix} = 1 - \beta - \beta\delta$$

The comparative statics w.r.t. G. [~~holding~~] [holding all others fixed]. ~~we have~~ are:

$$\begin{bmatrix} 1 & -1 & 0 \\ -\beta & 1 & \beta \\ -\delta & 0 & 1 \end{bmatrix} \begin{bmatrix} \partial \bar{Y}/\partial G_0 \\ \partial \bar{C}/\partial G_0 \\ \partial \bar{T}/\partial G_0 \end{bmatrix}$$

$$\therefore \frac{\partial \bar{Y}}{\partial G_0} = \frac{|J_{G_0}^{Y}|}{|J|} = \frac{\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{vmatrix}}{|J|}$$

$$= \frac{1}{1-\beta+\beta\delta}.$$