

Infinite Series of real numbers

Mathematics Sem 2, CC 3

Definition: Let $\{x_n\}$ be sequence of real numbers then $x_1 + x_2 + x_3 + \dots$ is a series of real numbers and is denoted by $\sum_{n=1}^{\infty} x_n$. We consider the sequence $\{s_n\}$, where $s_n = x_1 + \dots + x_n$, called the sequence of partial sum corresponding to the series $\sum_{n=1}^{\infty} x_n$. A series $\sum_{n=1}^{\infty} x_n$ is said to be convergent if the corresponding sequence of partial sums $\{s_n\}$ is convergent. The sum of the series $\sum_{n=1}^{\infty} x_n$ is $s = \lim_{n \rightarrow \infty} s_n$.

Problem: Test the convergence of the following series:

i) $1 + x + x^2 + \dots$

ii) $1 - 1 + 1 - 1 + \dots$

iii) $1 + 1 + 1 + \dots$

iv) $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots$

Solution: i) $s_n = 1 + x + \dots + x^n = \frac{1-x^{n+1}}{1-x}$.

Therefore $\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{1-x^{n+1}}{1-x} = \begin{cases} \frac{1}{1-x} & \text{for } |x| < 1 \\ \text{does not exist} & \text{for } |x| > 1 \end{cases}$.

Hence the series $1 + x + x^2 + \dots$ is convergent for $|x| < 1$ and divergent for $|x| \geq 1$.

ii) $s_1 = 1, s_2 = 0, s_3 = 1, s_4 = 0, \dots$

In general $s_{2n} = 0$ and $s_{2n+1} = 1 \forall n$. Hence $\{s_n\}$ is not convergent and consequently the given series is not convergent.

iii) Here $s_n = n \forall n$ and so $\lim_{n \rightarrow \infty} s_n$ does not exist finitely. Hence the given series is not convergent.

iv) Here $s_n = \frac{1}{1 \times 2} + \dots + \frac{1}{n(n+1)} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{1} - \frac{1}{n+1}$

Therefore $\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1}\right) = 1$.

Hence the given series is convergent.

Theorem: Cauchy's general principle of convergence for infinite series: A series $\sum_{n=1}^{\infty} x_n$ is convergent if and only if for any $\epsilon > 0$ there exists a positive integer K such that $|x_{n+1} + \dots + x_{n+p}| < \epsilon \quad \forall n \geq K, p = 1, 2, 3, \dots$

Problem: Test the convergence of the series $1 - 1 + 1 - 1 + \dots$.

Hint: $|x_{n+1} + x_{n+2}| = 2 \not< \frac{1}{2}$ for any n . Hence by Cauchy's general principle of convergence the given series is not convergent.

Problem: Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ & $\sum_{n=1}^{\infty} \frac{1}{2^n}$ are convergent but the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent.

Solution: Left as exercise.

Note: The series $\sum_{n=1}^{\infty} \frac{1}{n^p}, p > 0$, called p-series, is convergent for $p > 1$ and divergent for $p \leq 1$.

Note: If a series $\sum_{n=1}^{\infty} x_n$ is convergent then $\lim_{n \rightarrow \infty} x_n = 0$. The converse is not true. For example, the series $\sum \frac{1}{n}$ is not convergent although $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$.

Theorem (Comparison test): Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series of positive real numbers such that $a_n \leq lb_n \quad \forall n \geq K, a \text{ positive integer and } l > 0, a \text{ real number}$. Then

- i) If $\sum_{n=1}^{\infty} b_n$ is convergent then $\sum_{n=1}^{\infty} a_n$ is also convergent.
- ii) If $\sum_{n=1}^{\infty} a_n$ is divergent then $\sum_{n=1}^{\infty} b_n$ is also divergent.

Theorem (limit form of comparison test): Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series of positive real numbers and $l > 0$ be a positive real number such that $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = l$. Then the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge or diverge together.

Problem: Test the convergence of the following series:

- i) $\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$
- ii) $\sum_{n=1}^{\infty} \frac{1}{2n+1}$
- iii) $\sum_{n=1}^{\infty} \frac{n}{n+1}$

Solution/Hint: i) Let $a_n = \frac{1}{n^2+n+1}$ and $b_n = \frac{1}{n^2}$. Then $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2+n+1} = 1 \neq 0$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent the given series is convergent (by comparison test).

ii) Let $a_n = \frac{1}{2n+1}$ and $b_n = \frac{1}{n}$. Then $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent the given series is divergent (by comparison test).

iii) Left as an exercise.

Theorem (D. Alembert's Ratio Test): Let $\sum_{n=1}^{\infty} a_n$ be a series of positive real numbers such that $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = l$.

i) If $l < 1$ then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

ii) If $l > 1$ then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Problem: Test the convergence of $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

Solution: Let $a_n = \frac{n}{2^n}$. Then $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{(n+1)2^n}{2^{n+1}n} = \frac{1}{2} < 1$.

So by D. Alembert's Ratio test the given series is convergent.

Problem: Test the convergence of $1 + \frac{1}{2} + \frac{1.3}{2.4} + \frac{1.3.5}{2.4.6} + \dots$

Solution: Left for the readers.

Theorem (Cauchy Root Test): Let $\sum_{n=1}^{\infty} a_n$ be a series of positive real numbers such that $\sqrt[n]{a_n} = l$.

i) If $l < 1$ then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

ii) If $l > 1$ then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Problem: Test the convergence of $\frac{1}{3} + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{7}\right)^3 + \dots$

Solution : The given series is $\sum_{n=1}^{\infty} a_n$, where $a_n = \left(\frac{n}{2n+1}\right)^n$.

Therefore, $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{n}{2n+1}\right)^n} = \lim_{n \rightarrow \infty} \frac{n}{2n+1} = \frac{1}{2} < 1$

So by Cauchy Root Test the given series is convergent.

Theorem (Kummer's Test): Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \frac{1}{b_n}$ be two series of positive real numbers and let $c_n = \frac{a_n}{a_{n+1}} b_n - b_{n+1}$.

If $\lim_{n \rightarrow \infty} C_n = l > 0$ then $\sum_{n=1}^{\infty} a_n$ is convergent.

If $\lim_{n \rightarrow \infty} C_n = l < 0$ and $\sum_{n=1}^{\infty} \frac{1}{b_n}$ is divergent then $\sum_{n=1}^{\infty} a_n$ is divergent.

Theorem (Gauss's Test): Let $\sum_{n=1}^{\infty} a_n$ be a series of positive real numbers and let $\frac{u_n}{u_{n+1}} = 1 + \frac{a}{n} + \frac{b_n}{n^p}$ where $p > 1$ and the sequence $\{b_n\}$ is bounded. Then,

$\sum u_n$ is convergent if $a > 1$, $\sum u_n$ is divergent if $a \leq 1$.

Theorem (Alternative form of Gauss's Test): Let $\sum_{n=1}^{\infty} a_n$ be a series of positive real numbers and let $\frac{u_n}{u_{n+1}} = 1 + \frac{a}{n} + O\left(\frac{1}{n^p}\right)$ where $p > 1$. Then,

$\sum u_n$ is convergent if $a > 1$, $\sum u_n$ is divergent if $a \leq 1$.

$O\left(\frac{1}{n^p}\right) = h(n) \Rightarrow$ the sequence $\{h(n).n^p\}$ is bounded.

Problem: Test the convergence of the following series:

$$1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3} \cdot \frac{4}{5}\right)^2 + \left(\frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7}\right)^2 + \dots$$

Solution: Let $u_n = \left(\frac{2}{3} \cdot \frac{4}{5} \cdot \dots \cdot \frac{2n-2}{2n-1}\right)^2$, $n \geq 2$.

Then $\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} + \frac{1}{4n^2} = 1 + \frac{a}{n} + \frac{b_n}{n^p}$ where $a = 1$ and $b_n = \frac{1}{4} \forall n$ and so $\{b_n\}$ is

a bounded sequence. Hence by Gauss's test the given series is divergent.

Problem: Test the convergence of the following series:

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2} \cdot \frac{3}{4}\right)^2 + \left(\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6}\right)^2 + \dots$$

Solution: Let $u_n = \left(\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \dots \frac{2n-1}{2n}\right)^2, n \geq 2$.

$$\begin{aligned} \text{Then } \frac{u_n}{u_{n+1}} &= \frac{(2n+2)^2}{(2n+1)^2} = \left(1 + \frac{1}{n}\right)^2 \left(1 + \frac{1}{2n}\right)^{-2} \\ &= \left(1 + \frac{2}{n} + \frac{1}{n^2}\right) \left[1 - \frac{1}{n} + O\left(\frac{1}{n^2}\right)\right] = 1 + \frac{1}{n} + O\left(\frac{1}{n^2}\right) \end{aligned}$$

Hence by Gauss's test the given series is convergent.

Alternating series:

Definition: A series in the form $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ that is $a_1 - a_2 + a_3 - a_4 + \dots$, where $a_n > 0$, is called an alternating series.

Theorem (Leibnitz's test): If a sequence $\{a_n\}$ of positive real numbers is monotonic decreasing and $\lim_{n \rightarrow \infty} a_n = 0$ then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is convergent.

Proof: Let $s_n = a_1 - a_2 + a_3 - a_4 + \dots + (-1)^{n+1} a_n$.

Then $s_{2n+2} - s_{2n} = a_{2n+1} - a_{2n+2} \geq 0 \quad \forall n \in \mathbb{N}$ (as $\{a_n\}$ is monotonic decreasing)

Hence $s_{2n+2} \geq s_{2n} \quad \forall n \in \mathbb{N}$ and so $\{s_{2n}\}$ is monotonic increasing.

Also $s_{2n+1} - s_{2n-1} = -a_{2n} + a_{2n+1} \leq 0 \quad \forall n \in \mathbb{N}$ (as $\{a_n\}$ is monotonic decreasing)

So $s_{2n+1} \leq s_{2n-1} \quad \forall n \in \mathbb{N}$ and hence $\{s_{2n+1}\}$ is monotonic decreasing.

Again $s_{2n} = a_1 - a_2 + a_3 - a_4 + \dots - a_{2n}$

$$= a_1 - (a_2 - a_3) - (a_4 - a_5) + \dots - a_{2n} < a_1.$$

Therefore the sequence $\{s_{2n}\}$ is convergent as it is monotonic increasing and bounded above.

Also, $s_{2n+1} = a_1 - a_2 + a_3 - a_4 + \dots + a_{2n+1}$

$$= (a_1 - a_2) + (a_3 - a_4) + \dots + a_{2n+1} > a_1 - a_2.$$

Therefore the sequence $\{s_{2n+1}\}$ is convergent as it is monotonic decreasing and bounded below.

Also $\lim_{n \rightarrow \infty} (s_{2n+1} - s_{2n}) = \lim_{n \rightarrow \infty} a_{2n+1} = 0.$

So $\{s_{2n}\}$ and $\{s_{2n+1}\}$ converges to same limit and hence the sequence $\{s_n\}$ is convergent. Consequently $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is convergent.

Problem: Test the convergence of the following series:

i) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

ii) $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots$

Solution/ Hints: i) Here $a_n = \frac{1}{n} \forall n$. So the sequence $\{a_n\}$ is monotonic decreasing sequence of positive real numbers. Hence by Leibnitz's test the given series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ that is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ is convergent.

ii) Here $a_n = \frac{1}{\sqrt{n}} \forall n$. So the sequence $\{a_n\}$ is monotonic decreasing sequence of positive real numbers. Hence by Leibnitz's test the given series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ that is $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots$ is convergent.

Absolute Convergence: a series $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Note: If a series is absolutely convergent then it is convergent. But the converse is not true (see the example below).

Note: The series $\sum_{n=1}^{\infty} a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ is convergent but not absolutely convergent but the series $\sum_{n=1}^{\infty} (a_n)^2$ is convergent.

Note: The series $\sum_{n=1}^{\infty} a_n = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots$ is absolutely convergent and hence convergent but $\sum_{n=1}^{\infty} (a_n)^2$ is not convergent.

References:

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2. An introduction to Analysis – Differential Calculus –Part I: R. K. Ghosh & K. C. Maity, ISBN: 81-7381-437-6.
