Series of real numbers

Mathematics CC 2/GE 2

Definition: Let $\{x_n\}$ be sequence of real numbers then $x_1 + x_2 + x_3 + \cdots$ is a series of real numbers and is denoted by $\sum_{n=1}^{\infty} x_n$. We consider the sequence $\{s_n\}$, where $s_n = x_1 + \cdots + x_n$, called the sequence of partial sum corresponding to the series $\sum_{n=1}^{\infty} x_n$. A series $\sum_{n=1}^{\infty} x_n$ is said to be convergent if the corresponding sequence of partial sums $\{s_n\}$ is convergent. The sum of the series $\sum_{n=1}^{\infty} x_n$ is $\mathrm{s} = \lim_{n\to\infty} s_n$.

Problem: Test the convergence of the following series:

i) $1 + x + x^2 + \cdots$

ii) $1 - 1 + 1 - 1 + \cdots$

iii) $1 + 1 + 1 + \cdots$

iv) $\frac{1}{1X2} + \frac{1}{2X3} + \frac{1}{3X4} + \cdots$

Solution: i) $s_n = 1 + x + \cdots + x^n = \frac{1-x^n}{1-x}$.

Therefore $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-x^n}{1-x} = \begin{cases} \frac{1}{1-x} & for\ |x| < 1 \\ does\ not\ exist\ for\ |x| > 1 \end{cases}$.

Hence the series $+x + x^2 + \cdots$ is convergent for $|x| < 1$ and divergent for $|x| \geq 1$.

ii) $s_1 = 1, s_2 = 0, s_3 = 1, s_4 = 0, \ldots$

In general $s_{2n} = 0 \; and \; s_{2n+1} = 1 \; \forall n$. Hence $\{s_n\}$ is not convergent and consequently the given series is not convergent.

iii) Here $s_n = n \;\; \forall n$ and so $\lim_{n\to\infty} s_n$ does not exist finitely. Hence the given series is not convergent.

iv)Here $s_n = \frac{1}{1X2} + \cdots + \frac{1}{n(n+1)} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{1} - \frac{1}{n+1}$

Therefore $\lim_{n\to\infty} S_n = \lim_{n\to\infty}(1 - \frac{1}{n+1}) = 1.$

Hence the given series is convergent.

Theorem: Cauchy's general principle of convergence for infinite series: A series $\sum_{n=1}^{\infty} x_n$ is convergent if and only if for any $\varepsilon > 0$ there exists a positive integer K such that $\left|x_{n+1} + \cdots + x_{n+p}\right| < \varepsilon \quad \forall n \geq K, p = 1,2,3, \ldots$

Problem: Test the convergence of the series $1 - 1 + 1 - 1 + \cdots$.

Hint: $|x_{n+1} + x_{n+2}| = 2 \not< \frac{1}{2} \;\; for \; any \; n$ .Hence by Cauchy's general principal of convergence the given series is not convergent.

Problem: Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ & $\sum_{n=1}^{\infty} \frac{1}{2^n}$ are convergent but the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent.

Solution: Left as exercise.

Theorem (Comparison test): Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series of positive real numbers such that $a_n \leq l b_n \;\; \forall n \geq K, a \; positive \; integer \; and \; l > 0, a \; real \; number$. Then

i) If $\sum_{n=1}^{\infty} b_n$ is convergent then $\sum_{n=1}^{\infty} a_n$ is also convergent.

ii) If $\sum_{n=1}^{\infty} a_n$ is divergent then $\sum_{n=1}^{\infty} b_n$ is also convergent.

Theorem (limit form of comparison test): Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series of positive real numbers and $l > 0$ be a positive real number such that $\lim_{n\to\infty} \frac{a_n}{b_n} = l$. Then the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge or diverge together.

Problem: Test the convergence of the following series:

i) $\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$

ii) $\sum_{n=1}^{\infty} \frac{1}{2n+1}$

iii) $\sum_{n=1}^{\infty} \frac{n}{n+1}$

Solution/Hint: i) Let $a_n = \frac{1}{n^2+n+1}$ *and* $b_n = \frac{1}{n^2}$. Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2+n+1} = 1 \neq 0$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent the given series is convergent (by comparison test).

ii) Let $a_n = \frac{1}{2n+1}$ *and* $b_n = \frac{1}{n}$. Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent the given series is divergent (by comparison test).

iii) Left as an exercise.

Theorem (D. Alembert's Ratio Test): Let $\sum_{n=1}^{\infty} a_n$ be a series of positive real numbers such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$.

i)If $l < 1$ then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

ii) If $l > 1$ then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Problem: Test the convergence of $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

Solution: Let $a_n = \frac{n}{2^n}$. Then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)2^n}{2^{n+1}n} = \frac{1}{2} < 1$.

So by D. Alembert's Ratio test the given series is convergent.

Problem: Test the convergence of $1 + \frac{1}{2} + \frac{1.3}{2.4} + \frac{1.3.5}{2.4.6} + \cdots$

Solution: Left for the readers.

Theorem (Cauchy Root Test): Let $\sum_{n=1}^{\infty} a_n$ be a series of positive real numbers such that $\sqrt[n]{a_n} = l$.

i)If $l < 1$ then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

ii) If $l > 1$ then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Problem: Test the convergence of $\frac{1}{3} + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{7}\right)^3 + \cdots$

Solution : The given series is $\sum_{n=1}^{\infty} a_n$, $where\ a_n = \left(\frac{n}{2n+1}\right)^n$.

Therefore, $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{2n+1}\right)^n} = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} < 1$

So by Cauchy Root Test the given series is convergent.

Theorem (Rabbe's Test): Let $\sum_{n=1}^{\infty} a_n$ be a series of positive real numbers such that $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = l$.

i)If $l > 1$ then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

ii) If $l < 1$ then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Problem: Test the convergence of the following series:

$$1 + \frac{1}{2}.\frac{1}{3} + \frac{1}{2}.\frac{3}{4}.\frac{1}{5} + \frac{1}{2}.\frac{3}{4}.\frac{5}{6}.\frac{1}{7} + \cdots \quad .$$

Solution/ Hints: Here $a_n = \frac{1.3.5....(2n-5)}{2.4.6....(2n-2)}.\frac{1}{(2n-1)} \quad for\ all\ n \geq 2.$

Therefore $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{3}{2} > 1.$

Hence by Rabbe's test the given series is convergent.

Alternating series:

Definition: A series in the form $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ that is $a_1 - a_2 + a_3 - a_4 + \cdots$, where $a_n > 0$, is called an alternating series.

Theorem (Leibnitz's test): If a sequence $\{a_n\}$ of positive real numbers is monotonic decreasing and $\lim_{n\to\infty} a_n = 0$ then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is convergent.

Problem: Test the convergence of the following series:

i) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

ii) $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$

Solution/ Hints: i) Here $a_n = \frac{1}{n}$ $\forall n$. So the sequence $\{a_n\}$ is monotonic decreasing sequence of positive real numbers. Hence by Leibnitz's test the given series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ that is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is convergent.

ii) Here $a_n = \frac{1}{\sqrt{n}}$ $\forall n$. So the sequence $\{a_n\}$ is monotonic decreasing sequence of positive real numbers. Hence by Leibnitz's test the given series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ that is $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$ is convergent.

Absolute Convergence: a series $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent if the series $\sum_{n=1}^{\infty}|a_n|$ is convergent.

Note: If a series is absolutely convergent the it is convergent. But the converse is not true (see the example below).

Note: The series $\sum_{n=1}^{\infty} a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is convergent but not absolutely but the series $\sum_{n=1}^{\infty}(a_n)^2$ is convergent.

Note: The series $\sum_{n=1}^{\infty} a_n = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$ is absolutely convergent and hence convergent but $\sum_{n=1}^{\infty}(a_n)^2$ is not convergent.

References:

1. **Introduction to Real Analysis: S. K. Mapa, ISBN: 81-87169-02-8**
2. **An introduction to Analysis – Differential Calculus –Part I: R. K. Ghosh & K. C. Maity, ISBN: 81-7381-437-6.**