Sets in $\mathbb{R}$

Study Materials and Assignments for Sem 2,CC 3

Duration: 3 Hours

Definition (Neighbourhood of a point): Let x be a real number and S be a subset of $\mathbb{R}$. S is said to be a neighbourhood of x if there exists an open interval I_x such that $x \in I_x \subseteq S$.

Note: If S is a neighbourhood of x then $x \in S$. Is the converse true?

Ans: No. For example the sets $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ & $[0,1] = \{x \in \mathbb{R}: 0 \leq x \leq 1\}$ contains 1 but none is a neighbourhood of 1 (Why?)

Result: If S is a neighbourhood of x and $S \subseteq T$ then T is also a neighbourhood of T.

Proof: Let S be a neighbourhood of x and $S \subseteq T$ then there exists an open interval I_x such that $x \in I_x \subseteq S \subseteq T$. Hence T is a neighbourhood of x.

Note: $(0,1) = \{x \in \mathbb{R}: 0 < x < 1\}$, $[0,1] = \{x \in \mathbb{R}: 0 \leq x \leq 1\}$ & $[0,1] \cup \{2,3\}$ are also neighbourhoods of $\frac{1}{2}$ (Why?).

Note: For any real number x and for any positive , $N_\delta(x) = (x - \delta, x + \delta)$ is a neighbourhood of x , called δ neighbourhood of x.

Note: Any open interval is a neighbourhood of all its points (why?). The set $\mathbb{R}$, the set of all real numbers, is a neighborhood of all real numbers(why?).

Definition: Let S be a subset of $\mathbb{R}$ and x be a real number. x is said to be an **interior point** of S if there exists a neighbourhood $N(x)$ of x such that $N(x) \subseteq S$.

x is said to be an **exterior point** of S if there exists a neighbourhood $N(x)$ of x such that $N(x) \cap S = \emptyset$.

x is said to be a **boundary point** of S if every neighbourhood of x contains at least one point of S and contains one point which is not a point of S.

Note: If x is an interior point of s then $x \in S$. If x is an exterior point of s then $x \notin S$. If x is a boundary point of s then x may or may not be point of S (why?).

Problem: Find the set of all interior points, exterior points and boundary points of the following sets: i) $\mathbb{R}$ ii) $(0,1]$ iii) $\mathbb{N}$ iv) $\left\{\frac{1}{n}: n \in \mathbb{N}\right\}$ v) $\mathbb{Q}$ vi) $\mathbb{R}\backslash\mathbb{Q}$.

Solution: Left as exercise.

Definition: A set S is said to be open if each point of S is an interior point of S.

Note: $\emptyset$ & R are open sets.

Theorem: Open interval (a,b) is an open set.

Proof: Left for the readers.

Question: Is the converse of above theorem true? Justify.

Answer: No. The set $(0,1) \cup (2,3)$ is open (why?) but not an open interval.

Theorem: Finite union of open set is open.

Proof: Let $S_1, \dots, S_k$ be k (finite) open sets. Let $x \in S_1 \cup \dots \cup S_k$. Then $x \in S_i$ $for\ some\ i \in \{1, \dots, k\}$. Then $\delta_i > 0$ such that $(x - \delta_i, x + \delta_i) \subseteq S_i \subseteq S_1 \cup \dots \cup S_k$. Hence x is an interior point of $S_1 \cup \dots \cup S_k$. Since x is arbitrary point of $S_1 \cup \dots \cup S_k$, each point of $S_1 \cup \dots \cup S_k$ is an interior point of $S_1 \cup \dots \cup S_k$. Hence $S_1 \cup \dots \cup S_k$ is open.

Theorem: Arbitrary union of open set is open.

Proof/ Hints: Let $\{S_\alpha: \alpha \in \Lambda\}$ be an arbitrary collection of open sets, where Λ being index set. Let $x \in \bigcup_{\alpha\in\Lambda} S_\alpha$ then $x \in S_\lambda$ $for\ some\ \lambda \in \Lambda$. Rest part of the proof is similar to the proof of previous theorem.

Theorem: Finite intersection of open sets is open.

Proof: Let $S_1, \dots, S_k$ be k (finite) open sets. If $S_1 \cap \dots \cap S_k = \emptyset$, nothing to prove.

So, we consider the case when $S_1 \cap \dots \cap S_k \neq \emptyset$. Let $x \in S_1 \cap \dots \cap S_k$. Then $x \in S_1, \dots, S_k$. Then there exists $\delta_i > 0$, $i = 1, \dots, k$ such that $N_{\delta_i}(x) = (x - \delta_i, x + \delta_i) \subseteq S_i\ for\ all\ i = 1, \dots, k$.

Let $\delta = min\{\delta_1, \dots, \delta_k\}$. Then $\delta > 0$ and $N_\delta(x) = (x - \delta, x + \delta) \subseteq (x - \delta_i, x + \delta_i) \subseteq S_i\ for\ all\ i = 1, \dots, k$. Hence there exists a neighbourhood $N_\delta(x)$ of x such that $N_\delta(x) \subseteq S_1 \cap \dots \cap S_k$. Hence x is an interior point of $S_1 \cap \dots \cap S_k$. Since x is an arbitrary point of $S_1 \cap \dots \cap S_k$, $S_1 \cap \dots \cap S_k$ is open.

Question: What can you say about arbitrary intersection?

Answer: Arbitrary intersection of open sets may not be open. For example, each $I_n = \left(-\frac{1}{n}, \frac{1}{n}\right), n \in \mathbb{N}$ is open but $\bigcap_{n=1}^{\infty} I_n = \bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$ which is not open (why?).

Definition: The set of all interior points of a set S is called interior of S and is denoted by S°.

Theorem: The interior of a set is an open set.

Proof: Proof is left for the readers.

Note1: Interior of a set S is the largest open set contained in S (Why?).

Note2: Union of all open sets contained in a set$S$ is the interior of the set S.

Problem: Find the interior of the sets $(0,1), [0,1], \{0,1\}, (0,1], \mathbb{R}, \mathbb{Q}, \mathbb{R}\setminus\mathbb{Q}, \mathbb{N}, \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$.