Ex. A particle of unit mass moves under a central force $\mu\left\{\frac{3a}{r^4} - \frac{(a^2-b^2)}{r^5}\right\}$ $(a>b)$ being projected at right angles to the radius vector at a distance $(a+b)$ with velocity $\sqrt{\mu}/a+b$. Find the orbit

Solution : The D.E. of the orbit

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2} = \frac{1}{h^2}\left[3au^2 - 2(a^2-b^2)u^3\right]$$

Multiplying both side with $2\frac{du}{d\theta}$, we get the first integral as

$$\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{1}{h^2}\left(2au^3 - (a^2-b^2)u^4\right) + k.$$

$$v^2 = \frac{h^2}{p^2} = \frac{\mu}{(a+b)^2} \quad \text{gives} \quad k = 0 \quad \text{since } u = \frac{1}{a+b}$$

Hence we have $\left(\frac{du}{d\theta}\right)^2 = 2au^3 - (a^2-b^2)u^4 - u^2$

Put $\frac{du}{d\theta} = \frac{-1}{r^2}\frac{dr}{d\theta}$ to get

$$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{2a}{r^3} - \frac{a^2-b^2}{r^4} - \frac{1}{r^2}$$

$$\text{or,} \quad \frac{dr}{d\theta} = \pm\sqrt{b^2-(a-r)^2}$$

$$\text{or,} \int \frac{-dr}{\sqrt{b^2-(a-r)^2}} = \int d\theta$$