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- Prove that $2^{1/3}$ is irrational.

Let q : $2^{1/3}$ is irrational

$\neg q$: $2^{1/3}$ is not irrational i.e rational

Let us suppose that $\neg q$ is true.

$\therefore \exists$ integers a and b such that

$2^{1/3} = \frac{a}{b}$ where $g(a, b) = 1$

$\therefore 2 = \frac{a^3}{b^3}$

or $a^3 = 2b^3$

$\Rightarrow a^3$ is even

$\Rightarrow a$ is even

i.e $a = 2k$ where $k \in \mathbb{Z}$

$\therefore 8k^3 = 2b^3$

or $b^3 = 4k^3$

$\Rightarrow b$ is even

$\therefore \gcd(a, b) \neq 1$ as 2 is common factor to both a & b

$\therefore \neg q$ is false

$\therefore q$ is true

$\Rightarrow 2^{1/3}$ is irrational.

p : there is ~~a~~ no positive integer n

s.t $n^2+n^3=100$

$\neg p$ be true.

i.e $\exists$ a +ve integer n s.t

$$n^3+n^2=100$$

$$\text{or } n^2(n+1)=100.$$

n^2 divides 100

$\exists$ k s.t $100 = k * n^2$

$$= (n+1) * n^2$$

$n^2 | 100$ $\quad n = (1, 2, 5, 10)$

For each +ve integer n in this set

K is not $1+n$.

Contradiction.

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- Prove that there are no solution in integers x and y to the eqn. $2x^2+5y^2=14$

Ans: Let p: there are no solution in integers x and y to the eqn. $2x^2+5y^2=14$

Then ~~def~~ $\neg p$: there are sol^n in integers x & y to the eqn. $2x^2+5y^2=14$

Let p be not true

i.e $\neg p$ is true

i.e $\exists$ sol^n in integers x & y to the eqn. $2x^2+5y^2=14$.

First for the sum of $2x^2$ & $5y^2$ be 14 (i.e even) and $2x^2$ is even $5y^2$ have to be even.

$\exists$ some integer K s.t $2x^2 = 2K$

$\Rightarrow x^2 = K$

$5y^2 = 2K$

$K = \frac{5}{2}y^2$ which is not an integer. has to be an

If this ~~is an~~ integer then

$y = 2z$ for some integer z

$\therefore 2x^2 + 5 \times 4z^2 = 14$

or $x^2 + 10z^2 = 7$

But $10z^2 > 7$ if $z \neq 0$

So we must have $z = 0$

$\therefore x^2 = 7$, a contradiction

$\therefore$ Our assumption is false

$\therefore$ p is true.

- By principle of induction prove that $7^n - 1$ is divisible by 6 for all integers $n \geq 0$

- Let $P(n)$: $7^n - 1$ is divisible by 6 for all integers $n \geq 0$

$n = 0$ $7^0 - 1 = 1 - 1 = 0$ divisible by 6

$n = 1$ $7^1 - 1 = 7 - 1 = 6$ divisible by 6

$\therefore$ $P(n)$ is true for $n = 0$ & $n = 1$.

Let $P(n)$ be true for $n = k$

i.e $7^k - 1$ is divisible by 6

i.e $7^k - 1 = 6m$ for $m \in Z$

Now $P(k+1)$

$= 7^{k+1} - 1$

$= 7 \cdot 7^k - 1$

$= 7(1+6m) - 1$

$= 6 + 6m$

$= 6(1+m)$ divisible by 6 for $m \in Z$

$\therefore$ $P(n)$ is true for $n = k+1$ whenever it is true for $n = k$.

$\therefore$ $P(n)$ is true for all integers $n \geq 0$

- Use principle of induction to prove that

$3 + 3.5 + 3.5^2 + \cdots + 3.5^n$

$= \frac{3}{4}(5^{n+1} - 1) \; \forall n \in Z^+$

Let $P(n)$ be the mathematical expression:

$3 + 3.5 + 3.5^2 + \cdots + 3.5^n$

$= \frac{3}{4}(5^n - 1)$

$\underline{n=0}$ $\quad$ LHS $= 3$

RHS $= 3$ $\qquad$ _/_/_

For $n=1$, LHS $= 3+3.5 = 18$

RHS $= \frac{3}{4}(25^{2}-1)$

$= \frac{3}{\cancel{4}} \times \cancel{24}^{6} = 18$

$\therefore$ LHS $=$ RHS

$\therefore$ $P(1)$ is true

Let $P(n)$ be true for some $n=k \in \mathbb{Z}^+$

i.e $3+3.5+3.5^2+ \cdots + 3.5^k$

$= \frac{3}{4}(5^{k+1}-1)$

Now $3+3.5+3.5^2+ \cdots + 3.5^k + 3.5^{k+1}$

$= \frac{3}{4}(5^{k+1}-1) + 3.5^{k+1}$

$= 3\left\{\frac{5^{k+1}}{4} + 5.5^k\right\} - \frac{3}{4}$

$= 3.5^{k+1}\left\{\frac{1}{4}+1\right\} - \frac{3}{4}$

$= \frac{3}{4} \times 5^k \times 21 - \frac{3}{4}$

$= \frac{3}{4} \times 5^k \times (5+16) - \frac{3}{4}$

$= \frac{3}{4} \times 5^{k+1} + 3 \times 5^k \times 4 - \frac{3}{4}$

$= \frac{3}{4}(5^{k+2} - 1)$

$\therefore$ $P(k+1)$ is true.

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- If n is odd $n(n^2-1)$ is divisible by 24.

~~Let $f(n) = n(n^2-1) = n(n-1)(n+1)$~~

~~$f(1) = 24$~~

Induction on odd nos. is easier when expressed as $P(n_0)$ is true, $P(n) \Rightarrow P(n+2)$

Let $f(n) = n(n^2-1)$ be divisible by 24.

2^2-1

8.3

$f(1) = 0$ is divisible by 24.

$$f(n+2) - f(n) = (n+2)\{(n+2)^2 - 1\} - n(n^2-1)$$

$$= (n+2)(n^2+4n+4-1) - n(n^2-1)$$

$$= n\{n^2+4n+3-n^2+1\} + 2(n^2+4n+3)$$

$$= n(4n+4) + 2(n^2+4n+3)$$

$$= 6n^2 + 12n + 6$$

$$= 6(n+1)^2$$

Since n is odd.

$(n+1)$ is even

i.e $(n+1)^2$ is multiple of 4

$\therefore f(n+2) - f(n)$ is multiple of 24

So is $f(n+2)$

- Between every two rational nos. there is an irrational no.

Ans: p: Between every two rational nos. there is an irrational no.

Let p be false.

Then $\neg p$ be true.

i.e between every two rational nos. there does not exist an irrational no.

Let a & b be two unequal rational nos.

Let $a = \frac{m}{n}$, $b = \frac{p}{q}$ $\quad mq > np$

& let $a > b$.

Then $a - b = \frac{m}{n} - \frac{p}{q}$

$$= \frac{mq - np}{nq}$$

$\because (mq - np) \geq 1$, $a - b \geq \frac{1}{nq}$

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~~b~~ a

~~a = 3~~

~~b = 2~~

$$a > b + \frac{1}{nq} > b$$

$$\cancel{a + \frac{1}{\sqrt{2}nq}} > \cancel{b + \frac{1}{nq} + \frac{1}{\sqrt{2}nq}}$$

We can construct $b + \frac{1}{\sqrt{2}nq}$ in between b and a i.e

$$b < b + \frac{1}{\sqrt{2}nq} < a$$

which is irrational.