25/10/21

Classical Mechanics

1. Classical mechanics & general properties of Matter → D. P. Roychowdhury + S. N. Maiti
2. Theoretical Mechanics → Spiegel.
3. An introduction to Mechanics – D. Kleppner + R. Kolenkow.
4. Mechanics – P. K. Srivastava.

Newtonian Mechanics: (1687)

Principia (Latin)

(1642 → 1727)

C.M.

Kinematics

↓

Mathematical description of motion without consideration of causes (Forces)

Dynamics

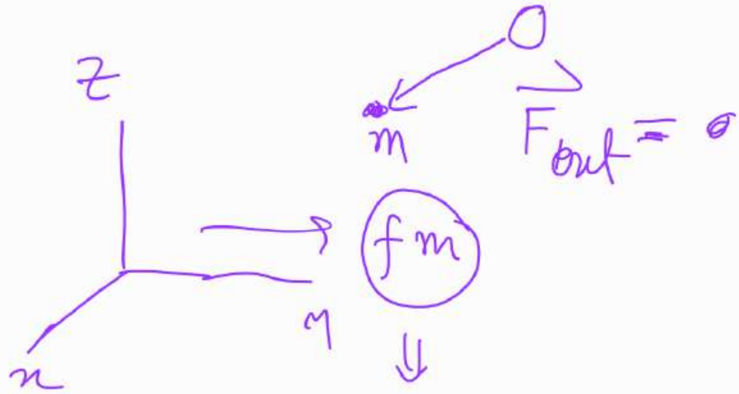
↓

Analysis the motion of objects subject to forces based on Newton Law.

1.

$F = mf \quad f = 0.$

$F = 0$



27/10/21

$$a = \frac{v^2}{R} = \omega^2 R = (0.73 \times 10^{-4} s^{-1})^2 (6.4 \times 10^8 cm)$$

$$\simeq 3.4 \ cm/s^2$$

Accⁿ due to orbital motion around the Sun

$$\omega_L = \frac{2\pi}{\pi \times 10^7} \simeq 2 \times 10^{-7}, \ R = 1.5 \times 10^{13} cm$$

$$a_L \approx \omega^2 R \approx 0.6 \ cm/s^2$$

Sun's accⁿ w.r.t. galactic centre

$$a_S = \omega^2 R = \frac{v^2}{R} = \frac{(3 \times 10^7)^2}{(3 \times 10^{22})} \simeq 3 \times 10^{-8} \ cm/s^2$$

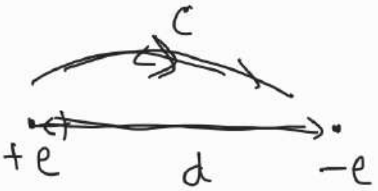
Interaction between two parts of the same system can not impart acceleration to the system as a whole. The interaction between horse & the cart can not accelerate the horse-cart system. You can not lift yourself by your hair.

Newton's 3rd law is not valid in e-m interaction. It is not satisfied by two separate charges interacting through e-m field. Propagation of electromagnetic field takes place at a finite speed 'c' - speed of light. The transfer of momentum i.e. travel of force field from one charge to other involves a time interval equal to distance between the charges divided by 'c'. So at an instant, a change in momentum occurs in first charge without an equal and opposite change in momentum in other charge i.e. instant by instant $\vec{F}_{12} \neq \vec{F}_{21} \Rightarrow \delta p_1 \neq \delta p_2$

$R = mg$

$R > mg$

$R < mg$



$t = \frac{d}{c}$

<u>Force</u>: An interaction that cause an acceleration of a body is called force.

<u>Mass</u>: It is a characteristic of a body that relates a force on the body to the resulting acceleration.

Newton's first law defines the criterion for an inertial frame. The Law says that there exists an absolute rest inertial frame w.r.t. which an object not acted upon by any external force, stays either at rest or in uniform motion along a straight line.

Daily forces of Physics:

03/11/21 Four fundamental interactions are in the Universe,

Gravity - Electromagnetic - Strong - Weak. First two forces are inverse square of distance.

The weak & strong forces have such short range that they are important only at nuclear distances $\sim 10^{-15}$ m.

They are negligible even at atomic distances $\sim 10^{-10}$ m

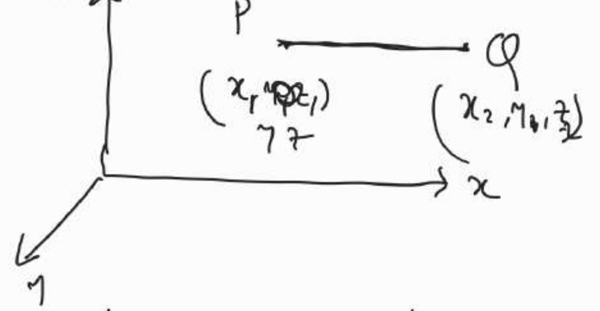
Between neutron-proton interchange weak force acts.

Scope of CM: The behaviours of all the bodies of linear size $> 10^{-6}$ m are adequately describable on class physics & velocity $< 10^{8}$ m/s.

Space-Time Symmetry & conservation law

Space homogeneous i.e. any physical process will occur the same way under identical conditions, no matter where it occurs & this leads to Newton's 3rd law, the consequence of which is conservation of linear momentum.

Points P, Q are equivalent for description of N's Law.



Let us apply the homogeneity property of space to an isolated system of two interacting particles; if both the particles are displaced by same amount $\delta\vec{r}$, there should be no change in the state of the system i.e. total work done by internal forces must be zero $\Rightarrow$

$$\underbrace{\vec{F}_{12}\cdot\delta\vec{r} + \vec{F}_{21}\cdot\delta\vec{r} = 0}\quad \vec{F}_{12} + \vec{F}_{21} = 0 \quad \text{as } \delta\vec{r} \text{ arbitrary}$$

$$\Rightarrow \vec{F}_{21} = -\vec{F}_{12} \longrightarrow \text{3rd Law}$$

$$\Rightarrow \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = \frac{d\vec{p}}{dt} = 0 \Rightarrow \vec{p} = \vec{p}_1 + \vec{p}_2 = \text{cst}$$

$\Rightarrow$ When two objects interact, the sum of linear momenta is cst in absence of external forces.

isolated system

This can be generalised for N-interacting particles:

$$\delta\vec{r}\cdot\sum_{i\neq j}\left(\vec{F}_{ij}+\vec{F}_{ji}\right)=0\,,\quad i,j=1\to N$$

Since space is homogeneous we can choose origin of our coordinate system anywhere

09/11/2021:

Time symmetry & energy conservation

Homogeneity of Time: It means one instant of time is identical to any other instant of time. Which implies, the forces of nature should not depend on time, e.g. force of gravitation between two masses separated by a distance remains the same for all time.

$$\frac{\partial\vec{F}}{\partial t}=0\,.$$

Now if force is conservative then we can write

$$F_x=-\frac{\partial U}{\partial x}\ \text{etc.}\ \Rightarrow U=U(x,y,z)$$ $\rightarrow$ should not depend on explicitly time 't'.

$$E = \frac{1}{2} m v^2 + U$$

$$\frac{dE}{dt} = m\vec{v}\cdot\frac{d\vec{v}}{dt} + \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{\partial t} + \frac{\partial U}{\partial z}\frac{dz}{dt}$$

$$= \vec{v}\cdot\vec{F} + (-F_x v_x - F_y v_y - F_z v_z)$$

$$= \vec{v}\cdot\vec{F} - \vec{F}\cdot\vec{v} = 0$$

E = cst. in time $\Rightarrow$ energy conservation

<u>Isotropy of space</u>: Means directional invariance means all directions from a definite point are equivalent. For two interacting particles (isolated), isotropy of space implies that net work done by internal forces must be zero when the system is rotated by an angle $\delta\varphi$

$$\Rightarrow d\vec{r}_1\cdot\vec{F}_{21} + d\vec{r}_2\cdot\vec{F}_{12} = 0$$

$$\because \vec{v} = \vec{\omega}\times\vec{r} \Rightarrow d\vec{r}_1 = \vec{\delta\varphi}\times\vec{r}_1 , \quad d\vec{r}_2 = \vec{\delta\varphi}\times\vec{r}_2$$

$$\Rightarrow (\vec{\delta\varphi}\times\vec{r}_1)\cdot\vec{F}_{21} + (\vec{\delta\varphi}\times\vec{r}_2)\cdot\vec{F}_{12} = 0$$

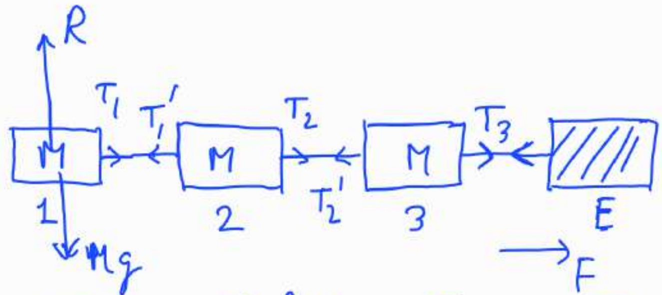
$$\Rightarrow \vec{\delta\varphi}\cdot(\vec{r}_1\times\vec{F}_{21}) + \vec{\delta\varphi}\cdot(\vec{r}_2\times\vec{F}_{12}) = 0$$

$$\Rightarrow \vec{r}_1\times\vec{F}_{21} + \vec{r}_2\times\vec{F}_{12} = 0$$

$$\Rightarrow \vec{\tau} = \frac{d\vec{L}}{dt} = 0$$ where $\vec{L} \to$ total ang. momentum for the system

$$\Rightarrow \vec{L} = \text{cst.}$$

Box Train;



Total mass = 3M, net force $\vec{F}$ is acting on it

Acceleration of the train $a = \frac{\vec{F}}{3M}$

For box-1 force T_1 is given by $T_1 = Ma = \frac{F}{3}$

$T_1 = T_1'$ (action reaction) in the rod between box 1, 2.

The net force on box-2,

$$T_2 - T_1' = Ma \Rightarrow T_2 = Ma + \frac{F}{3} = \frac{2F}{3}$$

The net force on box-3,

$$T_3 = T_2' + Ma = F$$

$\Rightarrow$

For N-box train net force $\frac{F}{N}$ $\quad a = \frac{F}{NM}$

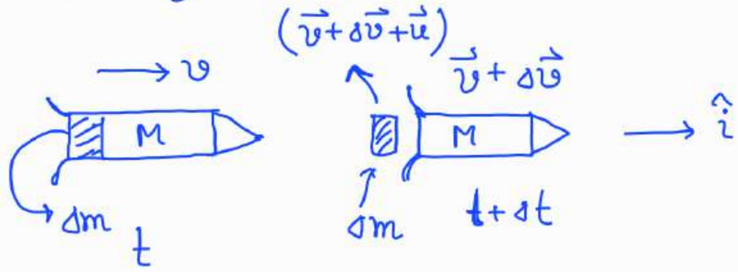
forward force on n-th box

$n\frac{F}{N}$

16/11/21

Rocket Motion

The motion of a Rocket is caused by the reaction force (thrust) produced by the ejection of fuel gas which forms a part of the rocket until it is ejected out. So, motion of rocket is equivalent to motion of a system with variable mass.



Let $\vec{v}$, $\vec{v}+\delta\vec{v}$ be the velocities of rocket at t & $t+\delta t$ relative to inertial frame, say ground.

The velocity of the mass of gas ejected from the backside of the rocket relative to the ground is

$\vec{v}+\delta\vec{v}+\vec{u}$, $(-\vec{u})$ is the velocity of the gas relative to rocket

$$\vec{P}_i = (M+\delta m)\vec{v} \qquad \vec{P}_f = M(\vec{v}+\delta\vec{v}) + \delta m(\vec{v}+\delta\vec{v}+\vec{u})$$

$$\delta\vec{P} = \vec{P}_f - \vec{P}_i = M\vec{v} + M\delta\vec{v} + \vec{v}\,\delta m + \underbrace{\delta m\,\delta\vec{v}}_{\simeq 0} + \vec{u}\,\delta m - M\vec{v} - \delta m\vec{v}$$

$$= M\delta\vec{v} + \delta m\,\vec{u}$$

$$\frac{d\vec{P}}{dt} = \lim_{\delta t\to 0}\frac{\delta\vec{P}}{\delta t} = M\frac{d\vec{v}}{dt} + \vec{u}\frac{dm}{dt} = M\frac{d\vec{v}}{dt} - \vec{u}\frac{dM}{dt} \quad \left[\because \frac{dm}{dt} = -\frac{dM}{dt}\right]$$

$$\therefore \vec{F} = M\frac{d\vec{v}}{dt} - \vec{u}\frac{dM}{dt}, \qquad \vec{F} \to \text{external force}$$

Eqn of motion of rocket

1. **Rocket in free space:** $\vec{F} = 0$

$$\Rightarrow \frac{d\vec{v}}{dt} = \frac{\vec{u}}{M}\frac{dM}{dt} \Rightarrow \int_{t_0}^{t_f} \frac{d\vec{v}}{dt}dt = \vec{u}\int_{t_0}^{t_f} \frac{1}{M}\frac{dM}{dt}dt$$

$$\Rightarrow \underline{\vec{v}_f - \vec{v}_0} = -\vec{u}\ln\left(\frac{M_0}{M_f}\right)$$

2. **Rocket in gravitational field:**

$\vec{g}\downarrow\ \uparrow$ $\downarrow\vec{u}$

17/11/21

$$\vec{F} = M\frac{d\vec{v}}{dt} - \vec{u}\frac{dM}{dt} = Mg$$

$\vec{u}, \vec{g}$ downward

$\uparrow\vec{v}$

If the fuel is ejected at const. rate β,

then $\frac{dm}{dt} = -\beta$

$$M(t) = M_0 - \beta t$$

As the fuel is ejected for a finite time τ till all of it is exhausted,

i.e. at $t = \tau$, $M(t) = M_s$, $M_s \to$ mass of space ship

$$\beta = M_F/\tau \Rightarrow M(t) = M_s + M_F\left(1 - \frac{t}{\tau}\right); \quad t \le \tau \qquad M(t) = M_0 = M_s + M_F$$

$$= M_s, \quad t > \tau$$

$$d\vec{v} = \vec{u}\frac{dM}{M} + g\,dt$$

$$\Rightarrow \vec{v}(t) = \vec{u}\ln M(t) + gt + C$$

at $t = 0$ $M(t) = M_0 = M_s + M_F$

$$\vec{v}(t) = \vec{v}_0 \quad , \quad c = \vec{v}_0 - \vec{u}\ln M_0$$

$$\vec{v}(t) = \vec{v}_0 + \vec{u}\ln\frac{M(t)}{M_0} + gt$$

$y \uparrow$

$$\Rightarrow \frac{d\vec{y}}{dt} = \vec{v}_0 + \vec{u}\ln\left(1 - \frac{\beta t}{M_0}\right) + gt$$

$$\Rightarrow |\vec{y}| = v_0 t + u\frac{M_0}{\beta}\left\{\left(1 - \frac{\beta t}{M_0}\right)\ln\left(1 - \frac{\beta t}{M_0}\right) - \left(1 - \frac{\beta t}{M_0}\right)\right\} + \frac{gt^2}{2} + D$$

at $t = 0$, $\quad y = 0$, $\quad c = u\frac{M_0}{\beta}$

$$\vec{v}_{max} = \vec{v}_0 + \vec{u}\ln\frac{M_S}{M_S + M_F} + g\tau \qquad \text{putting } t = \tau$$

One dimensional eq$^{\underline{n}}$ of motion

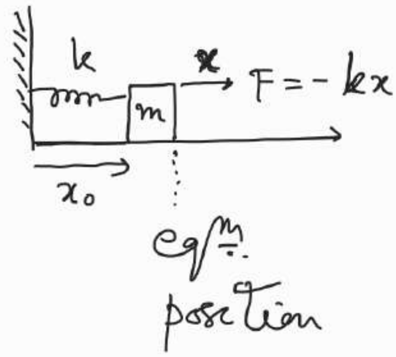
$$m\frac{d^2x}{dt^2} = F(x) \Rightarrow m\frac{dv}{dt} = F(x)$$

$$\Rightarrow m\int_{x_1}^{x_2}\frac{dv}{dt}\,dx = \int_{x_1}^{x_2}F(x)\,dx$$

$$\Rightarrow m\int_{t_1}^{t_2}\frac{dv}{dt}\,v\,dt = m\int_{t_1}^{t_2}\frac{d}{dt}\left(\frac{1}{2}v^2\right)dt = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

$$\Rightarrow \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = \int_{x_1}^{x_2}F(x)\,dx$$

Application: A block of mass 'm' is attached to one end of a horizontal spring, the other end is fixed. Block rests on horizontal frictionless surface



Write down the possible.

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = -k\int_{x_0}^{x} x\,dx = -\frac{1}{2}kx^2 + \frac{1}{2}kx_0^2$$

At $t=0$, $v_0 = 0$, mass is at a distance x_0 from origin

$$v^2 = -\frac{k}{m}x^2 + \frac{k}{m}x_0^2 \Rightarrow \frac{dx}{dt} = \sqrt{\frac{k}{m}}\sqrt{x_0^2 - x^2}$$

$$\int_{x_0}^{x} \frac{dx}{\sqrt{x_0^2 - x^2}} = \sqrt{\frac{k}{m}}\int_0^t dt = \sqrt{\frac{k}{m}}\,t = \sin^{-1}\left(\frac{x}{x_0}\right)\Big|_{x_0}^{x}$$

$$\Rightarrow \omega t = \sin^{-1}\left(\frac{x}{x_0}\right) - \pi/2 \quad \left(\omega = \sqrt{\frac{k}{m}}\right)$$

$$\Rightarrow x = x_0 \sin(\omega t + \pi/2)$$

Galilean invariance of Newton's Law & nature of force

Suppose a particle of mass 'm' moves with acceleration 'a' due to external force $\vec{F}$ as observed from S (inertial)

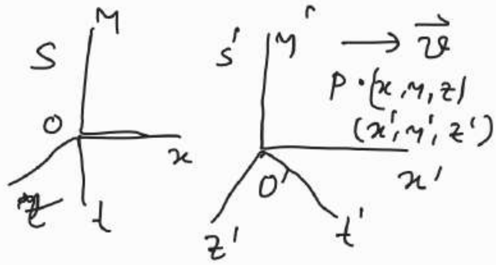
$$\vec{F} = m\vec{a}$$

Principle of relativity demands that for another inertial

S' , then $\vec{F}' = m\vec{a}'$

$\vec{a} = \vec{a}'$ as acceleration is invariant in G.T.

$x' = x - vt$



$\vec{F} = \vec{F}'$

$\vec{F}_{21}$ = force exerted on particle-1 by p-2 as observed from s

$$\begin{cases} x' = x - vt \\ y' = y \quad z' = z' , \; t' = t \end{cases}$$

$$\vec{F}_{21} = f(\vec{r}_2 - \vec{r}_1 , \; \vec{v}_2 - \vec{v}_1)$$

$$\vec{F}'_{21} = f(\vec{r}_2{}' - \vec{r}_1{}' , \; \vec{v}_2{}' - \vec{v}_1{}') = F_{21} \quad \text{as under G.T.}$$

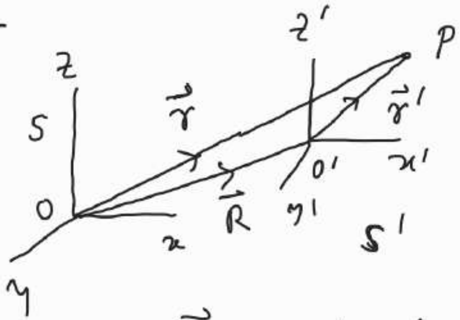
$$\vec{r}_2{}' - \vec{r}_1{}' = \vec{r}_2 - \vec{r}_1 \quad , \; \vec{v}_2{}' - \vec{v}_1{}' = \vec{v}_2 - \vec{v}_1$$

Further $\vec{F}_{12} = f(\vec{r}_1 - \vec{r}_2 , \; \vec{v}_1 - \vec{v}_2)$

$$= -f(\vec{r}_2 - \vec{r}_1 , \; \vec{v}_2 - \vec{v}_1) = -\vec{F}_{21}$$

$\rightarrow$ statement of 3rd law

Motion in non-inertial frame



$$\vec{r}(t) = \vec{r}'(t) + \vec{R}(t)$$

$$\dot{\vec{r}} = \dot{\vec{r}}' + \dot{\vec{R}}$$

$$\vec{v} = \vec{v}' + \vec{V} \Rightarrow \vec{a} = \vec{a}' + \vec{A}$$

$\vec{V} \rightarrow$ vel. of O' w.r.t. O

$\vec{A} = \dot{\vec{V}} =$ acc^n of O' w.r.t. O

$\vec{F} = m\vec{a}$ force acts on 'm' w.r.t. S

$\vec{F}' = m\vec{a}'$ .... S'

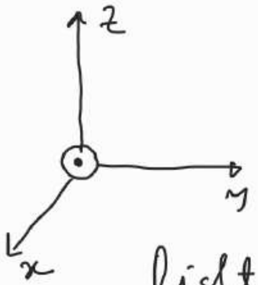
$\vec{F}_p = -m\vec{A}$ ... . S' due to acc^n of S'

$$m\vec{a} = m\vec{a}' + m\vec{A}$$

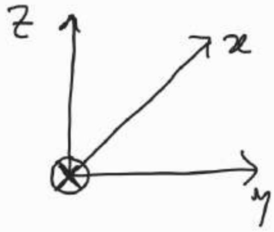
$$\Rightarrow \vec{F} = \vec{F}' + \vec{F}_p \Rightarrow \vec{F}' = \vec{F} - \vec{F}_p = m\vec{a} - m\vec{A}$$

pseudo force

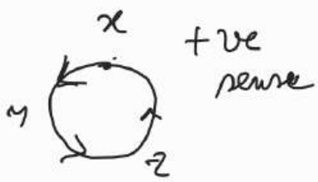
defined as force arising due to acceleration of a frame w.r.t. which the motion is considered.



Right handed



Left handed



$\odot$

an axis comes out back to front

$\otimes$

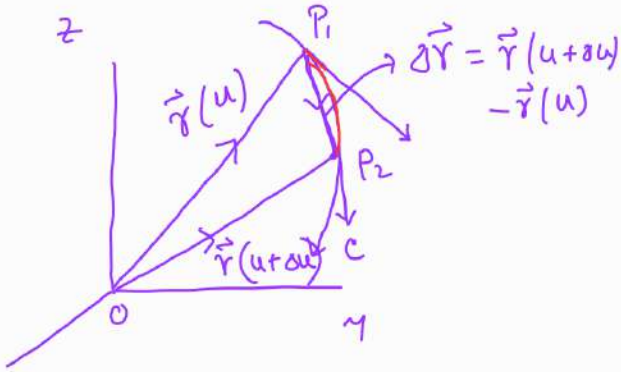
an axis goes front to back

Mechanics of single particle

Concept of particle: The concept of particle is that it is a mass point following a trajectory and whose size, internal structure, are negligible for the problem., A body need not be 'small' in size in the usual sense to be treated as particle. The size of the planets, sun being negligible compared to distance between them, their motion may be studied by treating them as particle.

23/11/21

Space Curve:



If c is a space curve defined by the function $\vec{r}(u)$

then $\vec{r}(u) = x(u)\hat{i} + y(u)\hat{j} + z(u)\hat{k}$

As parameter 'u' changes the position vector $\vec{r}$ describes a space curve having parametric eqn:

$$x = x(u),\ y = y(u),\ z = z(u)$$

Then $\lim_{\delta u \to 0} \frac{\delta\vec{r}}{\delta u} = \frac{d\vec{r}}{du} \to$ vector in the direction of tangent to the space curve at (x, y, z).

Velocity of the particle at P_1, $\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{du}\frac{du}{dt}$

$$= \frac{d\vec{r}}{ds}\frac{ds}{dt}, \quad \left|\frac{d\vec{r}}{du}\right| = \frac{\text{chord } \overline{P_1P_2}}{\text{arc } \widehat{P_1P_2}}$$

where 'u' is taken as arc length, 's' measured from P_1

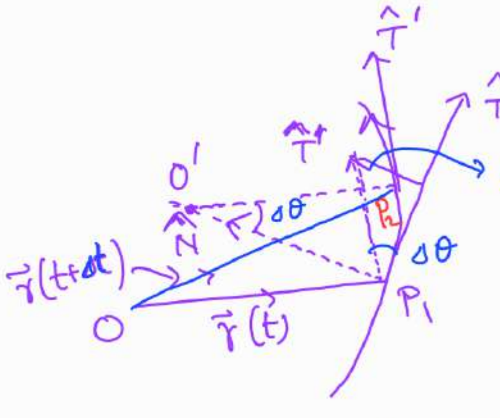
$$\left|\frac{d\vec{r}}{ds}\right| \to \frac{\text{chord } \overline{P_1P_2}}{\text{arc } \widehat{P_1P_2}} \to \frac{\delta r}{\delta s} \to \text{unity as } P_2 \text{ approaches}$$

P_1 & direction is that of tangent at P_1 in the forward direction, so

$$\frac{d\vec{r}}{ds} = \hat{T} \to \text{unit tangent at } P_1$$

$$\therefore \vec{v} = v\hat{T}$$

Acceleration at P_1 $\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv}{dt}\hat{T} + v\frac{d\hat{T}}{dt}$



$$= \frac{dv}{dt}\hat{T} + v\frac{d\hat{T}}{ds}\frac{ds}{dt}$$

as the direction of tangent depends on arc length 's'.

Considering two tangents $\hat{T}$ & $\hat{T}'$ at P_1 & P_2 and as P_2 approaches P_1 when $\delta t \to 0$, $\delta\hat{T}$ becomes $\perp^r$ both the unit vectors, two unit normals at P_1 & P_2 meet at O'

that approaches centre of curvature. Unit vector at P_1 along the normal towards O' is $\hat{N}$. Obviously $\Delta\hat{T} \parallel \hat{N}$ as $\Delta t \to 0$

$$\therefore \Delta\hat{T} = |\hat{T}| \sin(\Delta\theta)\, \hat{N} = \hat{N}\,\Delta\theta$$

$$\widehat{P_1P_2} = \Delta s = \rho\,\Delta\theta \qquad \rho = \overline{O'P_1} \rightarrow \text{radius of curvature}$$

$$\lim_{\Delta s \to 0} \frac{\Delta\hat{T}}{\Delta s} = \frac{d\hat{T}}{ds} = \frac{\hat{N}}{\rho}$$

$$\vec{a} = \frac{dv}{dt}\hat{T} + \frac{v^2}{\rho}\hat{N}$$

$\frac{dv}{dt}\hat{T} \rightarrow$ tangential acceleration

$\frac{v^2}{\rho}\hat{N} \rightarrow$ normal component of accⁿ

For a motion along straight line ($\rho \to \infty$)

$$\vec{a} = \dot{v}\hat{T}$$

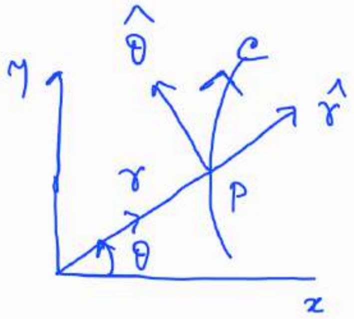
$$\frac{d\hat{T}}{ds} = k\hat{N} \qquad k = \frac{1}{\rho} \rightarrow \text{curvature at } P_1$$

A unit vector $\hat{B} = \hat{T} \times \hat{N}$ is called binormal to the curve at P_1

$\hat{B}, \hat{T}, \hat{N}$ may be used as coordinate system called moving triad.

Radial & Transverse component of velocity & acceleration in Polar Coordinate

$\vec{r} = x\hat{i} + y\hat{j} \qquad x = r\cos\theta \quad y = r\sin\theta$



$$\hat{r} = \frac{\partial\vec{r}}{\partial r} \Big/ \left|\frac{\partial\vec{r}}{\partial r}\right| \qquad \hat{\theta} = \frac{\partial\vec{r}}{\partial\theta} \Big/ \left|\frac{\partial\vec{r}}{\partial\theta}\right|$$

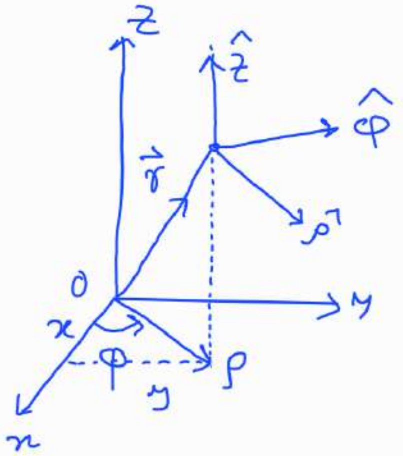
$$= \hat{i}\cos\theta + \hat{j}\sin\theta \qquad = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\dot{\hat{r}} = \frac{d\hat{r}}{dt} = -\hat{i}\sin\theta\,\dot{\theta} + \hat{j}\cos\theta\,\dot{\theta} = \dot{\theta}\hat{\theta} \;,\quad \dot{\hat{\theta}} = -\dot{\theta}\hat{r}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{r}) = \dot{r}\hat{r} + \dot{\hat{r}}r = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\vec{a} = \frac{d}{dt}(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

Cylindrical coordinate



$$\left.\begin{aligned} x &= \rho\cos\varphi \\ y &= \rho\sin\varphi \\ z &= z \end{aligned}\right\}$$ Cartesian to cylindrical trans^{m} eq^{n}

$$\vec{r} = \rho\hat{\rho} + z\hat{z}$$

$$= \rho\cos\varphi\,\hat{i} + \rho\sin\varphi\,\hat{j} + z\hat{k} \qquad \rho = (x^2+y^2)^{1/2}$$

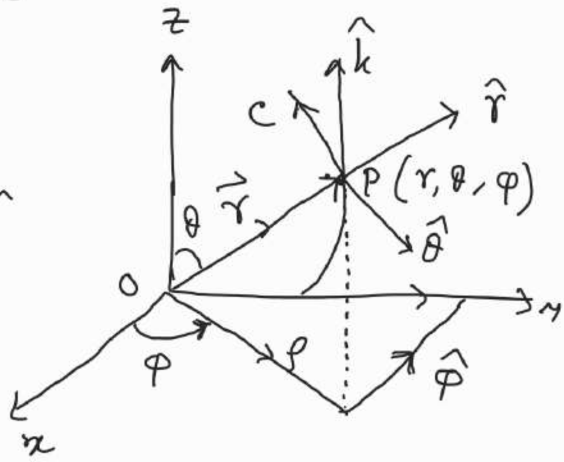
$$\varphi = \tan^{-1}\frac{y}{x}$$

$$\hat{\rho} = \cos\varphi\,\hat{i} + \sin\varphi\,\hat{j} \qquad \hat{\varphi} = -\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{\vec{r}} = \dot{\rho}\hat{\rho} + \rho\dot{\varphi}\hat{\varphi} + \dot{z}\hat{z}$$

$$\vec{a} = (\ddot{\rho} - \rho\dot{\varphi}^2)\hat{\rho} + (\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})\hat{\varphi} + \ddot{z}\hat{z}$$

24/11/21 : Velocity & acceleration in spherical polar coordinate



$$\left.\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta \end{aligned}\right\} \text{transformation eq}^{\underline{n}}\text{s}$$

$$\rho = r\sin\theta$$

$$r = (x^2+y^2+z^2)^{1/2}$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}$$

$$\varphi = \tan^{-1}\frac{y}{x}$$

$$\hat{r} = \hat{\rho}\sin\theta + \hat{z}\cos\theta = \hat{i}\cos\varphi\sin\theta + \hat{j}\sin\theta\sin\varphi + \hat{k}\cos\theta$$

$$\hookrightarrow \frac{\partial\vec{r}}{\partial r}\Big/\left|\frac{\partial\vec{r}}{\partial r}\right| \qquad \vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$$

$$= r\sin\theta\cos\varphi\,\hat{i} + r\sin\theta\sin\varphi\,\hat{j} + r\cos\theta\,\hat{k}$$

$$\hat{\theta} = \frac{\partial\vec{r}}{\partial\theta}\Big/\left|\frac{\partial\vec{r}}{\partial\theta}\right| = \hat{i}\cos\theta\cos\varphi + \hat{j}\cos\theta\sin\varphi - \hat{k}\cos\theta$$

$$\hat{\varphi} = \frac{\partial\vec{r}}{\partial\varphi}\Big/\left|\frac{\partial\vec{r}}{\partial\varphi}\right| = -\hat{i}\sin\varphi + \hat{j}\cos\varphi$$

$$\dot{\hat{r}} = \frac{\partial\hat{r}}{\partial t} = \underline{\dot{\theta}\,\hat{\theta} + \hat{\varphi}\,\dot{\varphi}\sin\theta}\,, \quad \dot{\hat{\theta}} = -\hat{r}\,\dot{\theta} + \cos\theta\,\dot{\varphi}\,\hat{\varphi}$$

$$\dot{\hat{\varphi}} = (\sin\theta\,\hat{r} - \cos\theta\,\hat{\theta})\,\dot{\varphi}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{r}) = \dot{r}\hat{r} + r\dot{\theta}\,\hat{\theta} + r\sin\theta\,\dot{\varphi}\,\hat{\varphi}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{r}\hat{r} + r\dot{\theta}\,\hat{\theta} + r\sin\theta\,\dot{\varphi}\,\hat{\varphi}\right)$$

$$= (\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\varphi}^2)\,\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\,\dot{\varphi}^2)\,\hat{\theta}$$

$$+ (r\sin\theta\,\ddot{\varphi} + 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta)\,\hat{\varphi}$$

Path integral of force = Work-energy theorem

If a force $\vec{F}(\vec{r})$ acting on a particle, is a function of position coordinate $\vec{r}$ then the work done by the force on the particle when it moves from $\vec{r}_1$ to $\vec{r}_2$, given by

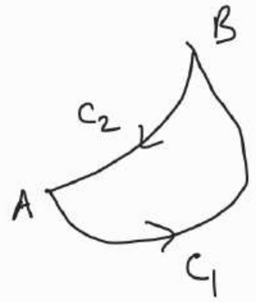
$$W_{12} = \int_{r_1}^{r_2} \vec{F}\cdot d\vec{r} = \int_{r_1}^{r_2} m\frac{d\vec{v}}{dt}\cdot d\vec{r} = \int_{r_1}^{r_2} m\frac{d\vec{v}}{dt}\cdot \vec{v}\,dt$$

$$= \int_{r_1}^{r_2} \frac{1}{2}m\,\frac{d}{dt}(v^2)\,dt = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

$$= T_2 - T_1 \rightarrow (1)$$

$$= \Delta E_k \rightarrow \text{change in K.E.}$$

Conservative force: If a line integral of force $\vec{F}$ taken around any closed path vanishes then the force is said to be conservative.



$$\Rightarrow \oint \vec{F}\cdot d\vec{r} = 0 \Rightarrow \int_{AC_1B} \vec{F}\cdot d\vec{r} + \int_{BC_2A} \vec{F}\cdot d\vec{r} = 0$$

$$\Rightarrow \int_{AC_1B} \vec{F}\cdot d\vec{r} = \int_{AC_2B} \vec{F}\cdot d\vec{r} \rightarrow \text{independent of path}$$

$$\Rightarrow \int_A^B \vec{F}\cdot d\vec{r} = -\left[V(r_B) - V(r_A)\right]$$

— ve sign to mean attractive force

$$= -\int_A^B dV(r) = -\int_A^B \vec{\nabla}V\cdot d\vec{r}$$

$$\Rightarrow \int_A^B (\vec{F} + \vec{\nabla}V)\cdot d\vec{r} = 0$$

$$\Rightarrow \quad \Rightarrow \quad \int_A^B \vec{F}\cdot d\vec{r} = V(r_A) - V(r_B)$$

$$= \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2 \quad (\text{eqn } 1)$$

$$\Rightarrow \quad \frac{1}{2}mv_A^2 + V(r_A) = \frac{1}{2}mv_B^2 + V(r_B)$$

$\Rightarrow (T+V)_A = (T+V)_B \Rightarrow T+V = \text{cst} \Rightarrow$ conservation of energy

Time integral of force:

$$\int_{t_1}^{t_2} \vec{F}\,dt = \int_{t_1}^{t_2} \frac{d\vec{p}}{dt}\,dt = \vec{p}_2 - \vec{p}_1 \rightarrow \text{change in momentum}$$

25/11/2021

System of Particles

Centre of Mass – If a system consists of 'n' no. of particles of masses m_i ($i = 1, \dots n$) having position vectors $\vec{r_i}$ w.r.t some arbitrary origin then the centre of mass 'G' for the system is defined as,

$$\vec{R} = \frac{m_1\vec{r_1} + m_2\vec{r_2} + \cdots}{m_1 + m_2 + \cdots} = \frac{\sum_i m_i \vec{r_i}}{\sum m_i} = \frac{1}{M}\sum_i m_i \vec{r_i}$$

$$= \left(\int_R \vec{r}\, dm\right) \Big/ \left(\int_R dm\right)$$

$$= \left(\int_R \rho \vec{r}\, dv\right) \Big/ \left(\int_R \rho\, dv\right)$$

$dm = \rho dv$

$\rho \to$ density

Example: Three masses m_1, m_2, m_3 (4 kg, 6 kg, 8 kg) have position vectors $\vec{r_1}(t, t^2, t^3)$, $\vec{r_2}(2t, 5t, 1)$, $\vec{r_3}(2, 4, 5)$. Find out the C.M., velocity of C.M. Linear momentum of the system, K.E. of the system at 5 sec.

$$C.M. = \vec{R} = \left[m_1\vec{r_1} + m_2\vec{r_2} + m_3\vec{r_3}\right] \Big/ (m_1 + m_2 + m_3)\Big]_{t = 5\text{ sec}}$$

$$= \left[\frac{4(t\hat{i} + t^2\hat{j} + t^3\hat{k}) + 6(2t\hat{i} + 5t\hat{j} + \hat{k}) + 8(2\hat{i} + 4\hat{j} + 5\hat{k})}{4+6+8}\right]_{t=5}$$

$$= \frac{1}{18}\left[(16t + 16)\hat{i} + (4t^2 + 30t + 32)\hat{j} + (4t^3 + 46\hat{k})\right]_{t=5}$$

$$= \frac{1}{18}\left[96\hat{i} + 282\hat{j} + 546\hat{k}\right]$$

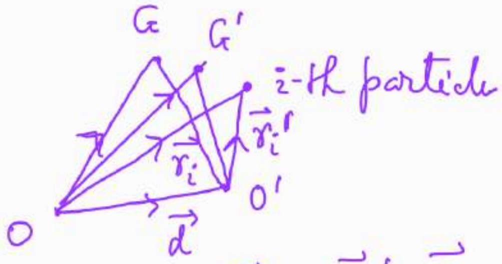
$\vec{V}_{CM} = \dot{\vec{R}}$ $\qquad$ $\vec{P}_{system} = \left(\sum_i m_i \vec{v}_i\right) / \left(\sum_i m_i\right)$

$\vec{v}_i = \dot{\vec{r}}_i$

$K.E. = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \frac{1}{2} m_3 v_3^2$

Theorem-I Show that C.M. is independent of origin.

If G is the C.M. w.r.t. origin 'O'



$$\vec{OG} = \frac{1}{M} \sum_i m_i \vec{r}_i$$

$\vec{r}_i = \vec{r}_i' + \vec{d}$

& if G' be C.M. w.r.t. origin O'

then $\vec{O'G'} = \frac{1}{M} \sum_i m_i \vec{r}_i' = \frac{1}{M} \sum_i m_i \vec{r}_i - \frac{1}{M} \sum_i m_i \vec{d}$

$= \vec{OG} - \vec{d}$

$\Rightarrow \vec{OG} = \vec{O'G'} + \vec{d} = \vec{OG'} \Rightarrow G = G'$

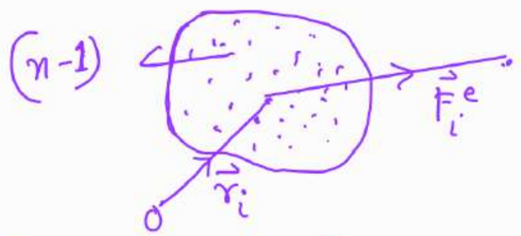
$\Rightarrow$ C.M. is independent of origin.

Theorem-2 The C.M. of a system of particles moves as if the total mass is concentrated at it & the net force is applied at that point.

Consider a system of particles & eqn. of motion for i-th particle,

$$\dot{\vec{p}}_i = \vec{F}_i^{(e)} + \sum_{j \neq i} \vec{F}_{ij} \;, \left[i = 1, \dots n\right] \rightarrow (1)$$

where $\vec{F}_i^{(e)}$ is the external force acting on i-th particle & $\vec{F}_{ij}$ is the internal force exerted on i-th particle by j-th one.



$$\sum_{j\neq i}\vec{F}_{ij} = \vec{F}_{i1}+\vec{F}_{i2}+\vec{F}_{i3}\cdots + (n-1) \text{ terms}$$

$\Downarrow$

$\vec{F}_{ii}$

$$\dot{\vec{p}}_i = \vec{F}_i^{(e)}$$

$$\dot{\vec{p}}_5 = \sum_{j=1\cdots 10}\vec{F}_{ij} = \vec{F}_{51}+\vec{F}_{52}+\cdots \vec{F}_{55}$$

$\Downarrow$ to be excluded

From 3rd law $\vec{F}_{ij} = -\vec{F}_{ji}$

Considering the system as a whole taking sum of eqn (i)

$$\sum_i \dot{\vec{p}}_i = \frac{d^2}{dt^2}\sum_i m_i\vec{r}_i = \sum_i \vec{F}_i^{(e)} + \sum_i\sum_{j\neq i}\vec{F}_{ij}$$

since $\displaystyle\sum_i\sum_{j\neq i}\vec{F}_{ij} = \sum_i{\sum_j}'\vec{F}_{ij} = \frac{1}{2}\sum_i{\sum_j}'\left[\vec{F}_{ij}+\vec{F}_{ji}\right]$

$= 0$ $\qquad \left[\vec{F}_{ij} = -\vec{F}_{ji}\right]$

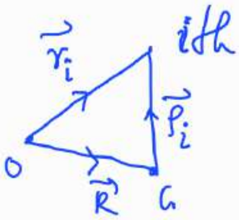
$$\sum_i \dot{\vec{p}}_i = \sum_i \vec{F}_i^{(e)} = \frac{d^2}{dt^2}\sum_i m_i\vec{r}_i = M\ddot{\vec{R}}$$

$$\left[\vec{R} = \frac{1}{M}\sum_i m_i\vec{r}_i\right]$$

$$\Rightarrow M\ddot{\vec{R}} = \sum_i \vec{F}_i^{(e)}$$

$\hookrightarrow$ eqn of motion of CM.

29/11/21 : **Theorem-3** : First moment of mass about C.M. vanishes or equivalently system of particles remain at rest w.r.t. C.M.



From defn. of C.M. $\vec{R} = \frac{1}{M}\sum_i m_i \vec{r}_i$

$M = \sum_i m_i$ $\qquad \vec{r}_i = \vec{R} + \vec{\rho}_i$

$$\Rightarrow \vec{R} = \frac{1}{M}\sum_i (\vec{R} + \vec{\rho}_i) m_i$$

$$= \vec{R} + \frac{1}{M}\sum_i m_i \vec{\rho}_i \Rightarrow \sum_i m_i \vec{\rho}_i = 0$$

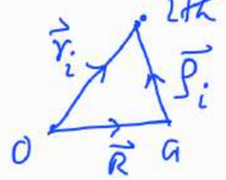
$$\vec{r}_i = \vec{R} + \vec{\rho}_i \Rightarrow \sum_i m_i \dot{\vec{r}}_i = \sum_i m_i (\dot{\vec{R}} + \dot{\vec{\rho}}_i) = M\dot{\vec{R}} + \sum_i m_i \dot{\vec{\rho}}_i$$

$$\Rightarrow M\dot{\vec{R}} = M\dot{\vec{R}} + \sum_i m_i \dot{\vec{\rho}}_i \Rightarrow \sum_i m_i \dot{\vec{\rho}}_i = 0$$

$\Downarrow$

System particles remains stationary w.r.t. C.M.

Theorem-4 K.E. of system of particles about any point equal to the K.E. of the C.M. assuming the total mass is located there plus K.E. of motion about C.M.



K.E. of the system of particles about any point 'O'

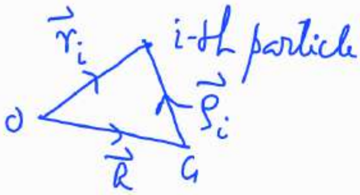
$\vec{r}_i = \vec{R} + \vec{\rho}_i$

$$T = \frac{1}{2}\sum_i m_i \dot{\vec{r}}_i^{\,2} = \frac{1}{2}\sum_i m_i (\dot{\vec{r}}_i \cdot \dot{\vec{r}}_i) = \frac{1}{2}\sum_i m_i (\dot{\vec{\rho}}_i + \dot{\vec{R}}) \cdot (\dot{\vec{\rho}}_i + \dot{\vec{R}})$$

$$= \frac{1}{2}\sum_i m_i (\dot{\vec{\rho}}_i^{\,2} + \dot{\vec{R}}^2 + 2\dot{\vec{\rho}}_i \cdot \dot{\vec{R}}) = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\sum_i m_i \dot{\vec{\rho}}_i^{\,2} + \dot{\vec{R}} \cdot \underbrace{\sum_i m_i \dot{\vec{\rho}}_i}_{0}$$

$$= \underbrace{\frac{1}{2}M\dot{\vec{R}}^2}_{\text{K.E. of motion of C.M.}} + \underbrace{\frac{1}{2}\sum_i m_i \dot{\vec{\rho}}_i^{\,2}}_{\text{K.E. of motion about C.M}}$$

Theorem. 5 : Angular momentum of system of particles about any point equal to the angular momentum of the total mass assumed to be located at C.M. plus the angular momentum about the C.M.



Ang. momentum of ith particle about the origin 'O'

$$\vec{r}_i = \vec{R} + \vec{\rho}_i$$

$$\vec{l}_i = \vec{r}_i \times \vec{p}_i = m_i (\vec{r}_i \times \vec{v}_i)$$

Total ang. momentum,

$$\vec{L} = \sum_i \vec{l}_i = \sum_i m_i (\vec{r}_i \times \vec{v}_i) = \sum_i m_i \left\{ (\vec{\rho}_i \times \vec{R}) \times (\dot{\vec{\rho}}_i + \dot{\vec{R}}) \right\}$$

$$= \sum_i m_i \left[(\vec{\rho}_i \times \dot{\vec{\rho}}_i) + (\vec{\rho}_i \times \dot{\vec{R}}) + (\vec{R} \times \dot{\vec{\rho}}_i) + (\vec{R} \times \dot{\vec{R}}) \right]$$

Now $\sum_i m_i \vec{R} \times \dot{\vec{\rho}}_i = \vec{R} \times \underbrace{\sum_i m_i \dot{\vec{\rho}}_i}_{=0} = 0$

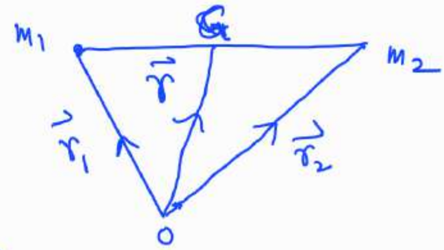
$$\Big(\underbrace{\sum_i m_i \vec{\rho}_i}_{=0}\Big) \times \dot{\vec{R}} = 0$$

$$\Rightarrow \vec{L} = \sum_i m_i (\vec{\rho}_i \times \dot{\vec{\rho}}_i) + M (\vec{R} \times \dot{\vec{R}}) = \underbrace{\sum_i \vec{\rho}_i \times (m_i \dot{\vec{\rho}}_i)}_{\vec{L}_{\text{w.r.t. C.M.}}} + \underbrace{\vec{R} \times (M \dot{\vec{R}})}_{\vec{L}_{CM}}$$

$$\vec{L}_{\text{w.r.t. C.M.}} + \vec{L}_{CM}$$

Theorem. 6 : Two particles having masses m_1 & m_2 move so that their relative velocity is $\vec{v}$ & velocity of C.M. is $\vec{V}$, then total energy of the system $T = \frac{1}{2} M V^2 + \frac{1}{2} \mu v^2$, $M = (m_1 + m_2)$, $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$

$$\vec{r} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$$



$$\vec{v}_1 = \dot{\vec{r}}_1 , \quad \vec{v}_2 = \dot{\vec{r}}_2 , \quad \dot{\vec{r}} = \vec{v}_1 - \vec{v}_2$$

$$\dot{\vec{r}} = \vec{V} \Rightarrow M\vec{V} = m_1\vec{v}_1 + m_2\vec{v}_2 \quad —(1)$$

$$\vec{v} = \dot{\vec{r}} = \vec{v}_1 - \vec{v}_2 \rightarrow (2)$$

Solving (1) & (2) $\quad \vec{v}_1 = \vec{V} + \dfrac{m_2\vec{v}}{m_1+m_2} = \vec{V} + \dfrac{\mu\vec{v}}{m_1}$

$$\vec{v}_2 = \vec{V} - \frac{m_1\vec{v}}{m_1+m_2} = \vec{V} - \frac{\mu\vec{v}}{m_2}$$

$$T = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 = \tfrac{1}{2}(m_1+m_2)^2V^2 + \tfrac{1}{2}\frac{m_1m_2}{m_1+m_2}v^2$$

Two body problem: Reduced mass

Consider the motion of two particles of masses m_1 & m_2 under the action of mutual action-reaction force. Show that the problem can be reduced to one as that of the motion of a single particle of mass $\mu = \dfrac{m_1m_2}{m_1+m_2}$

$$m_1\ddot{\vec{r}}_1 = \vec{F}_{1e} + \vec{F}_{12} \qquad m_2\ddot{\vec{r}}_2 = \vec{F}_{2e} + \vec{F}_{21} = \vec{F}_{2e} - \vec{F}_{12} \quad \left[\vec{F}_{21} = -\vec{F}_{12}\right]$$

$$\Rightarrow m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2 = \vec{F}_{1e} + \vec{F}_{2e}$$

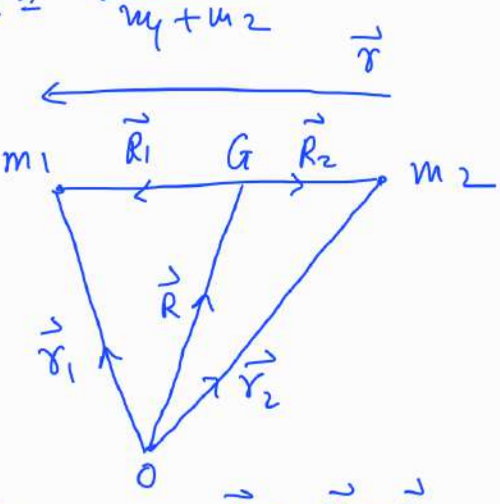
$$M\vec{R} = (m_1\vec{r}_1 + m_2\vec{r}_2) \Rightarrow M\ddot{\vec{R}} = m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2 = \vec{F}_{1e} + \vec{F}_{2e}$$

Introducing relative coordinate $\vec{r} = \vec{r}_1 - \vec{r}_2$

$$\ddot{\vec{r}} = \ddot{\vec{r}}_1 - \ddot{\vec{r}}_2 = \left(\frac{\vec{F}_{1e}}{m_1} - \frac{\vec{F}_{2e}}{m_2}\right) + \vec{F}_{12}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)$$

When the external forces are proportional to their masses and reside at large distances compared to mutual separation of m_1/m_2 so that $\vec{F}_{1e}$ & $\vec{F}_{2e}$ are almost $\parallel$ then 1st term vanishes.

$$\mu\ddot{\vec{r}} = \vec{F}_{12} \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}$$



$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

$$= \frac{m_1\vec{R}_1 + m_1\vec{R} + m_2\vec{R}_2 + m_2\vec{R}}{m_1 + m_2}$$

$$\Rightarrow m_1\vec{R}_1 + m_2\vec{R}_2 = 0$$

$$m_2\vec{R}_1 = m_2\vec{r}_1 - m_2\vec{R}$$

$$(m_1 + m_2)\vec{R}_1 = -m_2\vec{R}_2 - m_2\vec{R} + m_2\vec{r}_1$$

$$= -m_2\vec{r}_2 + m_2\vec{r}_1 = m_2\vec{r}$$

$$\Rightarrow \ddot{\vec{R}}_1 = \frac{m_2}{M}\ddot{\vec{r}} = \frac{\mu}{m_1}\ddot{\vec{r}} \Rightarrow \underline{\mu\ddot{\vec{r}} = m_1\ddot{\vec{R}}_1 = \vec{F}_{12}}$$

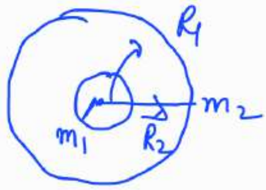
Similarly $m_2\ddot{\vec{R}}_2 = -\mu\ddot{\vec{r}} = -\vec{F}_{12} = \vec{F}_{21}$

$$\vec{R}_1 = \vec{r}_1 - \vec{R}$$
$$\vec{R}_2 = \vec{r}_2 - \vec{R}$$
$$\vec{r} = \vec{R}_1 - \vec{R}_2 = \vec{r}_1 - \vec{r}_2$$

Binary Star Problem

Two particles of masses m_1, m_2 at a distance $\vec{r}$ apart rotate about their common C.M. under the influence of gravity. Assuming their orbits to be circular find the time period of rotation & compare the angular momentum of the particles about C.M & K.E. of them.

If $\vec{R_1}, \vec{R_2}$ are the position of m_1, m_2 from the C.M. Then



$$\left|m_1 \vec{R_1}\right| = \left|m_2 \vec{R_2}\right|$$

From eqn of motion about CM.

$$m_1 \ddot{\vec{R_1}} = \frac{G m_1 m_2}{r^2} = m_1 \omega_1^2 R_1$$

$$m_2 \ddot{R_2} = \frac{G m_1 m_2}{r^2} = m_2 \omega_2^2 R_2$$

ω_1 / ω_2 ang. vel of m_1/m_2

$r \rightarrow$ relative separation of m_1, m_2

$$\Rightarrow \quad \omega_1 = \omega_2 = \omega \quad [m_1 R_1 = m_2 R_2]$$

$$\frac{G m_1 m_2}{r^2} = \frac{\omega^2 R_1}{\frac{1}{m_1}} = \frac{\omega^2 R_2}{\frac{1}{m_2}} = \frac{\omega^2 (R_1 + R_2)}{\frac{1}{m_1} + \frac{1}{m_2}} = \mu \omega^2 r$$

$$\Rightarrow \omega = \left[\frac{G(m_1 + m_2)}{r^3}\right]^{1/2} \Rightarrow T = \frac{2\pi}{\omega} = 2\pi \left[\frac{r^3}{G(m_1 + m_2)}\right]^{1/2}$$

$$\frac{l_2}{l_1} = \frac{m_2 v_2 R_2}{m_1 v_1 R_1} = \frac{m_2 \omega R_2^2}{m_1 \omega R_1^2} = \frac{m_1}{m_2} \quad [m_1 R_1 = m_2 R_2]$$

$$\frac{T_2}{T_1} = \frac{\frac{1}{2} m_2 v_2^2}{\frac{1}{2} m_1 v_1^2} = \frac{m_1}{m_2}$$

Energy conservation principle for system of particles

Consider a system of particles and eqn of motion of i-th particle

$$\dot{\vec{p}}_i = \frac{d\vec{p}_i}{dt} = \vec{F}_{ie} + \sum_{j\neq i} \vec{F}_{ij} \rightarrow (1)$$

Multiplying both sides by $\vec{v}_i$ velocity of i-th particle and summing over i-th

$$\sum_i m_i \frac{d\vec{v}_i}{dt} \cdot \vec{v}_i = \sum_i \vec{F}_{ie} \cdot \frac{d\vec{r}_i}{dt} + \sum_i \sum_{j\neq i} \vec{F}_{ij} \cdot \frac{d\vec{r}_i}{dt} \rightarrow (2)$$

$$\text{L.H.S.} = \sum_i m_i \frac{d\vec{v}_i}{dt} \cdot \vec{v}_i = \frac{d}{dt}\left(\sum_i \frac{1}{2} m_i v_i^2\right) = \frac{dT}{dt}, \quad T = \frac{1}{2}\sum_i m_i v_i^2$$

If $\vec{F}_{ie}$ & $\vec{F}_{ij}$ are both conservative then

$$\vec{F}_{ie} = -\vec{\nabla}_i V_{ie} \qquad \vec{F}_{ij} = -\vec{\nabla}_i V_{ij} \qquad [V_{ij} = V_{ji}]$$

$$\text{R.H.S} = -\sum_i \vec{\nabla}_i V_{ie} \cdot \frac{d\vec{r}_i}{dt} - \sum_i {\sum_j}' \nabla_i V_{ij} \cdot \frac{d\vec{r}_i}{dt}$$

$$= -\sum_i \left(\frac{\partial V_{ie}}{\partial x_i} \frac{dx_i}{dt} + \frac{\partial V_{ie}}{\partial y_i} \frac{dy_i}{dt} + \frac{\partial V_{ie}}{\partial z_i} \frac{dz_i}{dt} \right)$$

$$-\frac{1}{2} \sum_i {\sum_j}' \left(\nabla_i V_{ij} \cdot \frac{d\vec{r}_i}{dt} + \nabla_j V_{ji} \cdot \frac{d\vec{r}_j}{dt} \right) \qquad [\vec{F}_{ij} = -\vec{F}_{ji}]$$

$$= -\frac{dV_{ie}}{dt} - \frac{1}{2} \sum_i {\sum_j}' \frac{dV_{ij}}{dt}$$

$$\Rightarrow (2) \rightarrow \frac{dT}{dt} = -\frac{dV_{ie}}{dt} - \frac{1}{2} \sum_i {\sum_j}' \frac{dV_{ij}}{dt}$$

$$\Rightarrow \frac{d}{dt}\left(T + V_{ie} + \frac{1}{2}\sum_i {\sum_j}' V_{ij}\right) = 0 \Rightarrow T + V' = \text{const}$$

$$V' = V_{ie} + \frac{1}{2}\sum_i {\sum_j}' V_{ij}$$

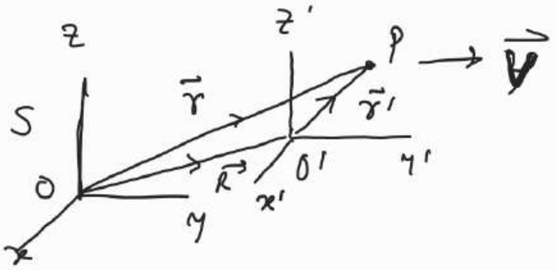
06/12/21

Reference frame

Inertial frame ⇓ frame either at rest or in uniform motion

Non Inertial frame ⇓ accelerated frame

Motion observed from accelerated frame : Pseudo force



$$\vec{r} = \vec{r}' + \vec{R}$$

$$\vec{v} = \vec{v}' + \vec{V}$$

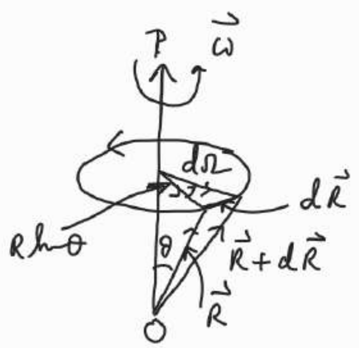
$$\vec{a} = \vec{a}' + \vec{f}_{s's}$$

$$m\vec{a} = \vec{F} = m\ddot{\vec{r}}' + m\ddot{\vec{R}}$$

Eq^n of motion as observed from s' frame

$$m\ddot{\vec{r}}' = \vec{F} - m\ddot{\vec{R}} \rightarrow \text{pseudo force}$$

Rotating frame of reference :



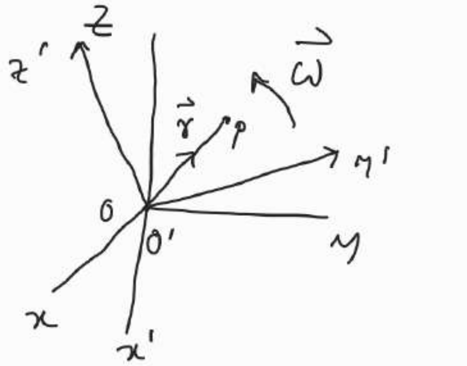
$$|d\vec{R}| = R\sin\theta\, d\Omega \qquad [\, s = r\theta$$

$$\Rightarrow d\vec{R} = d\vec{\Omega} \times \vec{R}$$

$$\Rightarrow \frac{d\vec{R}}{dt} = \frac{d\vec{\Omega}}{dt} \times \vec{R} = \vec{\omega} \times \vec{R}$$

$$\Rightarrow \frac{d(\,)}{dt} \equiv \vec{\omega} \times (\,)$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$= x'\hat{i}' + y'\hat{j}' + z'\hat{k}'$$



$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left(x'\hat{i}' + y'\hat{j}' + z'\hat{k}'\right)$$
$$= \left.\frac{\partial\vec{r}}{\partial t}\right|_{\hat{i}',\hat{j}',\hat{k}' \text{ fixed}} + x'\left(\frac{d\hat{i}'}{dt}\right) + y'\left(\frac{d\hat{j}'}{dt}\right) + z'\left(\frac{d\hat{k}'}{dt}\right)$$

$$\frac{d()}{dt} = \vec{\omega}\times()$$
$$\frac{d\hat{i}'}{dt} = \vec{\omega}\times\hat{i}'$$

$$= \frac{\partial\vec{r}}{\partial t} + x'\left(\vec{\omega}\times\hat{i}'\right) + y'\left(\vec{\omega}\times\hat{j}'\right) + z'\left(\vec{\omega}\times\hat{k}'\right)$$
$$= \frac{\partial\vec{r}}{\partial t} + \vec{\omega}\times\vec{r}$$

$$\Rightarrow \frac{d()}{dt} \equiv \frac{\partial()}{\partial t} + \vec{\omega}\times() \qquad \Rightarrow \frac{d}{dt} = \frac{d'}{dt} + \vec{\omega}\times$$

$$\Rightarrow \frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left(\frac{d'\vec{r}}{dt} + \vec{\omega}\times\vec{r}\right) = \left(\frac{d'}{dt} + \vec{\omega}\times\right)\left(\frac{d'\vec{r}}{dt} + \vec{\omega}\times\vec{r}\right)$$

acceleration w.r.t. fixed frame

$$= \frac{d'^2\vec{r}}{dt^2} + \frac{d'}{dt}(\vec{\omega}\times\vec{r}) + \vec{\omega}\times\frac{d'\vec{r}}{dt} + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$
$$= \frac{d'^2\vec{r}}{dt^2} + \frac{d'\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}\times\frac{d'\vec{r}}{dt} + \vec{\omega}\times\frac{d'\vec{r}}{dt} + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$
$$= \frac{d'^2\vec{r}}{dt^2} + 2\vec{\omega}\times\frac{d'\vec{r}}{dt} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \frac{d'\vec{\omega}}{dt}\times\vec{r}$$

$\Downarrow$ acc^n w.r.t rotating S' frame

$\Downarrow$ Coriolis acceleration

$\Downarrow$ Centripetal acc^n (centrifugal acc^n)

$\Downarrow$ 0 if $\vec{\omega}$ const

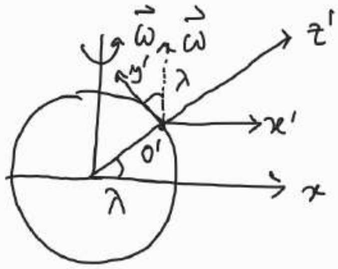
$\Rightarrow$ Eq^n of motion for a particle of mass 'm' as observed from rotating frame S' is

$$m\frac{d'^2\vec{r}}{dt^2} = m\frac{d^2\vec{r}}{dt^2} - 2m\vec{\omega}\times\frac{d'\vec{r}}{dt} - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) - m\frac{d'\vec{\omega}}{dt}\times\vec{r}$$

$\Downarrow$

Coriolis force

Deviation of freely falling bodies from vertical



Let x', y', z' be the axes along horizontally eastward, horizontally northward and vertically upward, angular velocity of earth $\vec{\omega}$ at point o' having latitude λ can be expressed as

$$\vec{\omega} = \omega \cos\lambda\, \hat{j}' + \omega \sin\lambda\, \hat{k}'$$

Let a particle of 'm' is freely falling with velocity $\vec{v}'$

given by $\vec{v}' = -v'\hat{k}'$ (z' being +ve along upward)

Coriolis force on the particle

$$\vec{F}_c = -2m\left(\vec{\omega} \times \frac{d'\vec{r}}{dt}\right)$$

$$= 2m\omega v' \cos\lambda\, \hat{i}' \rightarrow \text{particle deflects along eastward}$$

Eqn of motion of the particle,

$$m\frac{d^2x'}{dt^2} = 2m\omega v' \cos\lambda$$

$$\Rightarrow \frac{d^2x'}{dt^2} = 2v'\omega\cos\lambda = 2gt\omega\cos\lambda \quad [v' = gt]$$

$$\Rightarrow \frac{dx}{dt} = 2g\omega\cos\lambda\, \frac{t^2}{2} + \text{const}$$

$$\text{at } t=0, \quad \frac{dx'}{dt} = 0 \quad \Rightarrow \text{const} = 0$$

$$\Rightarrow \frac{dx}{dt} = g\omega\cos\lambda\, t^2 \Rightarrow x' = \frac{1}{3}g\omega\cos\lambda\, t^3 + \text{const}$$

$$\text{const} = 0, \quad x' = 0 \text{ at } t = 0$$

$$x = \frac{1}{3} g\omega \cos\lambda\, t^3 \qquad h = \frac{1}{2} g t^2 = \text{height attained by the particle at time } 't'$$

$$= \frac{2}{3} \omega h \cos\lambda \sqrt{\frac{2h}{g}} \qquad t = \sqrt{\frac{2h}{g}}$$

Effect of Coriolis force :

$\vec{F}_c = -2m\omega \times \vec{v}'$, $\vec{v}'$ velocity of a particle in s' frame

$\vec{\omega} \to$ ang velocity of the non-inertial frame w.r.t. fixed frame s

So it acts on a particle moving in non-inertial frame.

In the northern hemisphere $\vec{F}_c$ tends to deflect a projectile along the earth's surface to the right of its direction of motion, and opposite in southern hemisphere.

Effect on wind circulation - cyclone

Coriolis force plays an important role on wind circulation.

Due to unequal heating

13/12/21

at equator & poles, there is always a pressure gradient from poles to equator which causes air masses to move toward equator.

If there is low pressure area locally, air rushes toward it from all sides which are deflected towards right in N-hemisphere. This makes cyclone move anticlockwise in N-hemisphere and clockwise in S-hemisphere.

A region of low pressure with roughly concentric isobars is known as cyclone. In absence of Coriolis force direction of motion of air is $\perp^r$ to isobars from high to low pressure but the wind circulates around high pressure zone in opposite sense.

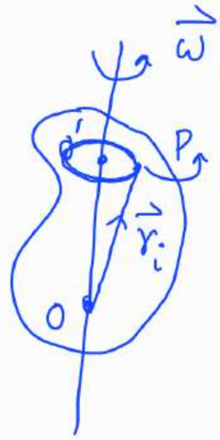
Rigid Body dynamics:

A system of particles in which the distance between any two of them remains fixed is called rigid body.

A rigid body can have two types of motion - translational & rotational..

A rigid body in motion can therefore be completely specified by the position of one point (CM) and its orientation w.r.t. that point.

So, a rigid body has 6 d.o.f., 3 for translational motion & 3 for rotational motion. A rigid body with one point fixed (translational motion is frozen) has only 3 d.o.f. for rotation. A rigid body with two points fixed can have one d.o.f. only as it can rotate about the axis passing through those two fixed points. A rigid body with three non-linear points fixed can have no motion.



Rotation about a fixed axis:

When any two points O, O' on the body are fixed, the rigid body only can rotate about OO' axis with angular velocity $\vec{\omega}$.

Angular momentum of the i-th particle about the point O,

$$\vec{l}_i = \vec{r}_i \times m_i \vec{v}_i \Rightarrow \vec{L} = \sum_i \vec{l}_i = \sum_i \vec{r}_i \times m_i (\vec{\omega} \times \vec{r}_i)$$

$$\vec{L} = \sum_i \left[m_i (\vec{r}_i \cdot \vec{r}_i) \vec{\omega} - (\vec{r}_i \cdot \vec{\omega}) \vec{r}_i \right] \longrightarrow (1)$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}$$

$$L_x = \sum_i m_i \left[r_i^2 \omega_x - (x_i \omega_x + y_i \omega_y + z_i \omega_z) x_i \right]$$

$$= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

$$I_{xx} = \sum_i m_i (r_i^2 - x_i^2) = \sum_i m_i (y_i^2 + z_i^2)$$

$\to$ M.I. about x axis

$$I_{xy} = I_{yx} = -\sum_i m_i x_i y_i \qquad I_{xz} = I_{zx} = -\sum_i m_i x_i z_i$$

$$L_y = \sum_i m_i \left[r_i^2 \omega_y - (x_i \omega_x + y_i \omega_y + z_i \omega_z) y_i \right]$$

$$= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$

$$I_{yx} = -\sum_i m_i x_i y_i \,, \quad I_{yy} = \sum_i m_i (r_i^2 - y_i^2)$$

$$= \sum_i m_i (x_i^2 + z_i^2)$$

$$I_{yz} = -\sum_i m_i y_i z_i$$

$$L_z = \sum_i m_i \left[r_i^2 \omega_z - (x_i \omega_x + y_i \omega_y + z_i \omega_z) z_i \right]$$

$$= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

$$I_{zz} = \sum_i m_i (r_i^2 - z_i^2) = \sum_i m_i (x_i^2 + y_i^2)$$

$I_{yx} = I_{xy}$, I_{yz}, I_{xz} $\to$ product of inertial about those axis

In matrix form eqn (i) would be

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

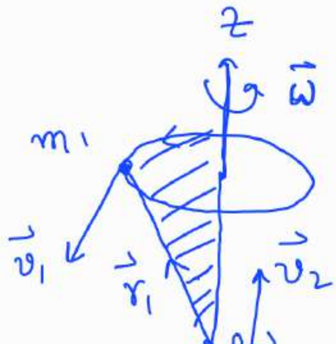
$$\Rightarrow \vec{L} = \mathbf{I}\,\vec{\omega} \qquad\qquad \vec{L} = I\,\vec{\omega}$$

$\Downarrow$ M.I. $\qquad\qquad\qquad \vec{a}\ \vec{b}$

$\to$ Tensor of 2nd rank $\qquad\qquad$ scalar M.I.

A mathematical structure having nine components in 3-Dimensions attached to a coordinate system if connected by means of linear transformation to the corresponding quantities in a different coordinate system derived from the first, then that is called a tensor of 2nd rank.

<u>$\vec{L}$ & $\vec{\omega}$ are not $\parallel$ in general</u>



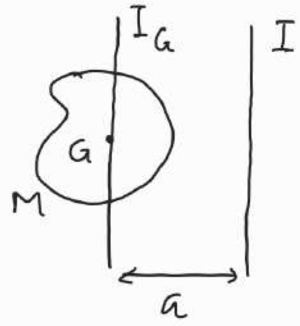
A system of two masses m_1, m_2 connected by a rigid rod (massless) rotates about z-axis with an angular velocity $\vec{\omega}$. The ang. momentum of the system

$\vec{L} = \vec{r}_1 \times m_1\vec{v}_1 + \vec{r}_2 \times m_2\vec{v}_2$

Thus $\vec{L}$ is a vector $\perp^r$ to the connecting rod & hence can not be $\parallel^l$ to $\vec{\omega}$. [As the particles rotate, the vector $\vec{L}$ which is $\perp^r$ to rod & lies in the shaded plane, also rotates & traces out a cone with the vertex at O.]

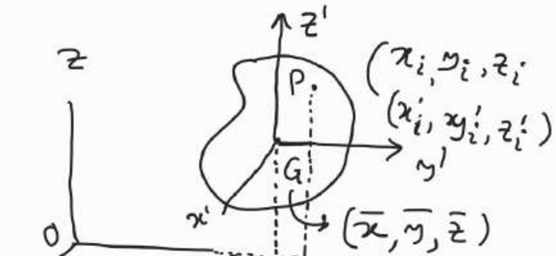
20/12/21

Parallel axis theorem



The moment of inertia (I) of a body about any axis is the sum of its M.I. (I_G) about an axis parallely passing through its C.M. (G) and the product of the mass (M) of the body by the square of the distance (a^2) between the two axes i.e

$$I = I_G + Ma^2.$$



Proof: Suppose G is the C.M. of the body having coordinates $(\bar{x}, \bar{y}, \bar{z})$ w.r.t. O of (x, y, z) system and (x_i, y_i, z_i) are the coordinates of i-th particle P.

Considering another coordinate system $(x'y'z')$ with origin G and axes $\parallel$ to those (xyz) system, the coordinates of the particle are (x_i', y_i', z_i')

$$\therefore \quad x_i = x_i' + \bar{x}, \quad y_i = \bar{y} + y_i', \quad z_i = \bar{z} + z_i'$$

The M.I. of the body about ox axis,

$$I_{xx} = \sum_i m_i (y_i^2 + z_i^2) = \sum_i m_i \{(\bar{y} + y_i')^2 + (\bar{z} + z_i')^2\}$$

$$= \sum_i m_i (\bar{y}^2 + \bar{z}^2) + \sum_i m_i (y_i'^2 + z_i'^2) + 2 \sum_i m_i \bar{y} y_i' + 2 \sum_i m_i z_i' \bar{z}$$

$$= (\bar{y}^2 + \bar{z}^2) M + I_G + 2\bar{y} \underbrace{\sum_i m_i y_i'}_{=0} + 2\bar{z} \underbrace{\sum_i m_i z_i'}_{=0}$$

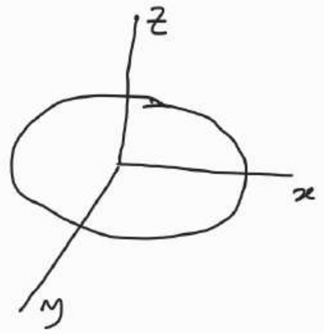
$$= Ma^2 + I_G$$

M.I. of the body about Gx' axis that passes through G & $\parallel$ to ox axis, $a = \sqrt{\bar{y}^2 + \bar{z}^2}$ is distance of C.M. 'G' from ox axis.

Perpendicular axis theorem

The sum of M.I. of a planar body about two mutually perpendicular axes in its plane is equal to the M.I. about an axis $\perp^r$ to the plane passing through the point of intersection of these two axes in the plane.

$$I_{xx} + I_{yy} = I_{zz}$$



Taking x-y axes in the plane of the body & z axis $\perp^r$ to the plane we have,

$$I_{xx} = \sum_i m_i (y_i^2 + z_i^2) = \sum_i m_i y_i^2 \quad (\text{all } z_i = 0 \text{ in } x\text{-}y \text{ plane})$$

$$I_{yy} = \sum_i m_i (x_i^2 + z_i^2) = \sum_i m_i x_i^2 \quad (\quad " \quad)$$

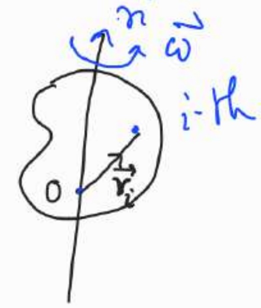
$$I_{zz} = \sum_i m_i (x_i^2 + y_i^2), \quad I_{xy} = I_{yx} = 0, \quad I_{yz} = I_{zx} = 0$$

$$I_{zz} = I_{xx} + I_{yy}$$

Radius of gyration.

The dimension of M.I. are ML^2 so that M.I. of a body can be equated to the product of the mass of the body and some appropriate length K i.e. $I = MK^2$, the quantity K is called radius gyration about the axis which means if we had placed a particle of same mass at a distance K from the given axis, the M.I. of the particle would be equal to that of the body.

Kinetic energy of a rotating body



Let us consider a rigid body, be capable of rotation about a given axis defined by unit vector $\hat{n}$.

K.E. of motion for the whole body

$$T = \frac{1}{2} \sum_i m_i v_i^2$$

$v_i \rightarrow$ linear velocity of i-th particle w.r.t. origin O

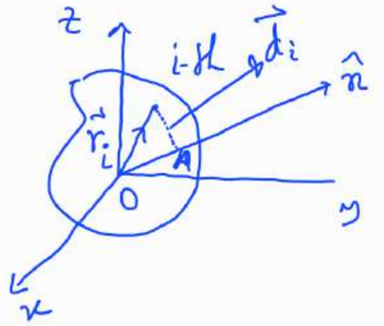
$$\Rightarrow T = \frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2}\sum_i m_i \vec{v}_i \cdot (\vec{\omega}\times\vec{r}_i)$$

$$= \frac{1}{2}\sum_i m_i \vec{\omega}\cdot(\vec{r}_i \times \vec{v}_i) = \frac{1}{2}\vec{\omega}\cdot\sum_i \vec{r}_i \times (m_i \vec{v}_i)$$

$$= \frac{1}{2}\vec{\omega}\cdot\vec{L} \qquad \vec{L} = \sum_i \vec{r}_i \times \vec{p}_i = \text{ang. momentum}$$

for the whole body w.r.t. 'O'

M.I. w.r.t. arbitrary axis

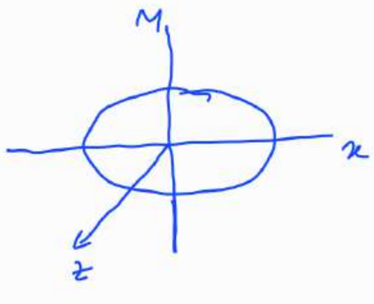


Suppose the M.I. and product of inertia of a rigid body w.r.t. x-y-z system are I_{xx}, I_{yy}, I_{zz}, I_{xy}, I_{yz}, I_{zx} etc are the product of inertia of the body, we are to find out the M.I. about an axis making an angle α, β, γ with x-y-z system.

$$I_{\hat{n}} = I_{11}\lambda^2 + I_{22}\mu^2 + I_{33}\nu^2 + 2I_{12}\lambda\mu + 2I_{23}\mu\nu + 2I_{13}\lambda\nu$$

$$= I_{11}\cos^2\alpha + I_{22}\cos^2\beta + I_{33}\cos^2\gamma + 2I_{12}\cos\alpha\cos\beta + 2I_{23}\cos\beta\cos\gamma + 2I_{13}\cos\alpha\cos\gamma$$

where (λ, μ, ν) or $(\cos\alpha, \cos\beta, \cos\gamma)$ are direction cosine of $\hat{n}$



If m_i has position vector $\vec{r}_i$ w.r.t. origin 'O', its moment of inertia about $\hat{n}$ is

$$m_i d_i^2 \quad , \quad d_i = |\vec{r}_i \times \hat{n}|$$

$$\lambda^2 + \mu^2 + \nu^2 = 1$$

$$\hat{n} = \lambda \hat{i} + \mu \hat{j} + \nu \hat{k}$$

Now $$\vec{r}_i \times \hat{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_i & y_i & z_i \\ \cos\alpha & \cos\beta & \cos\gamma \end{vmatrix}$$

$$|\vec{r}_i \times \hat{n}| = (y_i^2 + z_i^2)\cos^2\alpha + (x_i^2 + z_i^2)\cos^2\beta + (x_i^2 + y_i^2)\cos^2\gamma$$
$$- 2x_i y_i \cos\alpha\cos\beta - 2x_i z_i \cos\alpha\cos\gamma - 2y_i z_i \cos\beta\cos\gamma$$

$$I_{\hat{n}} = \sum_i m_i d_i^2 = \sum_i m_i (\vec{r}_i \times \hat{n})^2$$

$$= I_{11}\cos^2\alpha + I_{22}\cos^2\beta + I_{33}\cos^2\gamma + 2I_{12}\cos\alpha\cos\beta + 2I_{23}\cos\beta\cos\gamma + 2I_{13}\cos\alpha\cos\gamma$$

where $I_{11} = \sum_i m_i (y_i^2 + z_i^2) \quad I_{22} = \sum_i m_i (x_i^2 + z_i^2)$

$I_{33} = \sum_i m_i (x_i^2 + y_i^2) \quad , \quad I_{12} = -\sum_i x_i y_i m_i$

$I_{23} = -\sum_i m_i y_i z_i \qquad I_{31} = -\sum_i m_i z_i x_i$

<u>Ellipsoid of inertia.</u>

When a rigid body rotates about fixed point 'O'

The instantaneous axis of rotation passing through the point 'O' changes its direction with time. The M.I. of the body about these different axes will in general be different. These different values of M.I. can be represented geometrically if one takes $1/\sqrt{I_{\hat{n}}}$ instead of $I_{\hat{n}}$. If the axis under consideration be $\hat{n}$ & $\vec{\rho}$ is a vector defined as

$\vec{\rho} = \dfrac{\hat{n}}{\sqrt{I_{\hat{n}}}}$ then the expression for M.I. of a body w.r.t. an axis $\hat{n}$ can be written as,

$$I_{11}x_1^2 + I_{22}x_2^2 + I_{33}x_3^2 + 2I_{12}x_1x_2 + 2I_{23}x_2x_3 + 2I_{31}x_1x_3 = 1 \quad \rightarrow (1)$$

$$I_{\hat{n}} = I_{11}\cos^2\alpha + I_{22}\cos^2\beta + I_{33}\cos^2\gamma + 2I_{12}\cos\alpha\cos\beta + 2I_{23}\cos\beta\cos\gamma + 2I_{13}\cos\alpha\cos\gamma$$

$$x_1^2 + x_2^2 + x_3^2 = \rho^2 = \frac{1}{I_{\hat{n}}} \qquad \begin{aligned} \rho_x &= x_1 = \rho\cos\alpha \\ \rho_y &= x_2 = \rho\cos\beta \\ \rho_z &= x_3 = \rho\cos\gamma \end{aligned}$$

For different values of d.c. $(\cos\alpha, \cos\beta, \cos\gamma)$ i.e for different axis $\hat{n}$ we have different values of M.I. about these axes passing through the fixed point 'O'. If we measure along each direction, a distance $\overline{OA} = \dfrac{1}{\sqrt{I_{\hat{n}}}}$, $I_{\hat{n}}$ is the M.I. about the axis $\hat{n}$, then the locus of A is found to be that type of eq^n given by (1) which is eq^n of an ellipsoid in terms of coordinates x_1, x_2, x_3.

If the coordinate axes are rotated to coincide with the PAI (Principal Axis of Inertia) of the ellipsoid the eqn (1) becomes,

$$I_1 x_1^2 + I_2 x_2^2 + I_3 x_3^2 = 1$$

where x_1, x_2, x_3 represent the coordinates of new axes.

The surface represented by eqn (1) is a closed surface ellipsoid in nature & hence called ellipsoid of inertia

PAI & PMI

For a body of any shape and mass distribution it is always possible to find a set of mutually $\perp^r$ axes fixed at a point in the body such that the product of inertia about these axes vanish. Such axes are called Principal axes. In other words, a set of axes w.r.t. which if a body rotates, the direction of angular momentum would be same as that of angular velocity i.e. $\vec{L} = I\vec{\omega}$, I would be scalar instead of 2nd rank tensor. The inertia tensor w.r.t. these principal axes obviously has a diagonal form and the diagonal elements of the inertia tensor is called Principal Moment of Inertia (PMI).

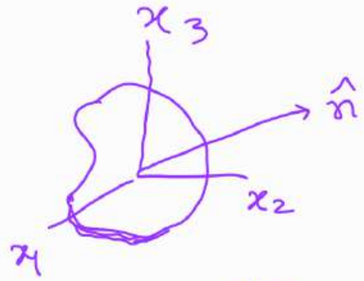
When all the PMI of a body are different ($I_1 \neq I_2 \neq I_3$) the body is asymmetrical top. If the two of them are equal ($I_1 = I_2 \neq I_3$), it is called symmetrical top. In this case the direction of one of the principal axis in the x_1-x_2 plane may be chosen arbitrarily. If all the PMI are equal ($I_1 = I_2 = I_3$) we have a spherical top.

28/12/21

M.I. of a body about a given axis can not be larger than the largest of the PMI or smaller than the smallest of PMI

or

Out of three PMI at a point one is the largest and another is the smallest of all the possible M.I. about axes passing through the point.



x_1, x_2, x_3
PMI

Proof: We know that the M.I. of a body about an axis defined by unit vector $\hat{n}$ having direction cosines $(\alpha_1, \alpha_2, \alpha_3)$ w.r.t. x-y-z system,

$$I_{\hat{n}} = I_{11}\alpha_1^2 + I_{22}\alpha_2^2 + I_{33}\alpha_3^2 + 2I_{12}\alpha_1\alpha_2 + 2I_{23}\alpha_2\alpha_3 + 2I_{31}\alpha_1\alpha_3$$

If $(\alpha_1', \alpha_2', \alpha_3')$ are the d.c. of the rotation axis w.r.t. PAI, then it reduces to

$$I_{\hat{n}}' = \alpha_1'^2 I_1 + \alpha_2'^2 I_2 + \alpha_3'^2 I_3 \qquad [\alpha_1'^2 + \alpha_2'^2 + \alpha_3'^2 = 1]$$

$$= (1 - \alpha_2'^2 - \alpha_3'^2) I_1 + \alpha_2'^2 I_2 + \alpha_3'^2 I_3$$

$$= I_1 - \alpha_2'^2 (I_1 - I_2) - \alpha_3'^2 (I_1 - I_3)$$

If $I_1 > I_2, I_3 \Rightarrow I_{\hat{n}}' < I_1$ & $I_{\hat{n}}' = I_1$ for rotation about x_1 axis $(\alpha_2' = 0 = \alpha_3')$

$I_1 < I_2, I_3 \Rightarrow I_{\hat{n}}' > I_1$ & $I_{\hat{n}}' = I_1$ for above reason

Prolate - oblate: With the principal axis transformation the general equation of ellipsoid of inertia

$$I_{11} x_1^2 + I_{22} x_2^2 + I_{33} x_3^2 + 2I_{12} x_1 x_2 + 2I_{23} x_2 x_3 + 2I_{31} x_1 x_3 = 1 \rightarrow (1)$$

can be converted always into standard canonical form w.r.t. those PAI with semi-axes,

$$a = 1/\sqrt{I_1} \quad b = 1/\sqrt{I_2} \, , \quad c = 1/\sqrt{I_3}$$

$$\Rightarrow \quad I_1 x_1^2 + I_2 x_2^2 + I_3 x_3^2 = 1 \Rightarrow \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1 \rightarrow (2)$$

If any two of the semi-axes say a, b are equal i.e. when $I_1 = I_2$ the surface is an ellipsoid of revolution & it is called oblate if $a = b > c$ or $I_1 = I_2 < I_3$ and called prolate when $a = b < c$ or $I_1 = I_2 > I_3$. When $a = b = c$ or $I_1 = I_2 = I_3$ the ellipsoid becomes sphere or spheroid of radius $1/\sqrt{I_1}$.

Determination of PMI & PAI

From the definition of PMI, $\vec{L} = I\vec{\omega}$ where I is simply scalar and hence in component form it becomes

$$L_1 = I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 = I\omega_1$$

$$L_2 = I_{21}\omega_1 + I_{22}\omega_2 + I_{23}\omega_3 = I\omega_2$$

$$L_3 = L_{31}\omega_1 + I_{32}\omega_2 + I_{33}\omega_3 = I\omega_3$$

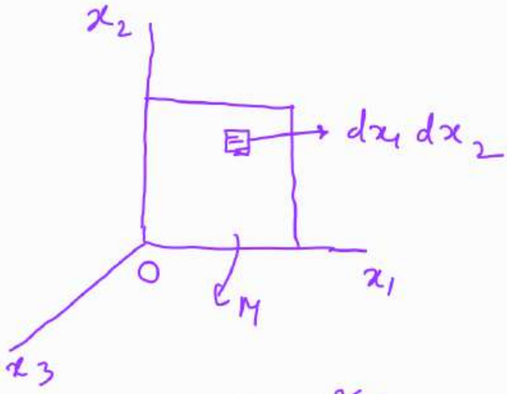
$$\Rightarrow \begin{pmatrix} I_{11}-I & I_{12} & I_{13} \\ I_{21} & I_{22}-I & I_{23} \\ I_{31} & I_{32} & I_{33}-I \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = 0$$

This characteristic eqn has non-trivial soln. for ω_i iff,

$$\det \left| I_{ij} - I\mathbb{1} \right| = 0 \rightarrow (2)$$

The cubic eqn in I has three real roots $I = I_1, I_2, I_3$ and called PMI. The direction of $\vec{\omega}$ i.e. direction of PAI can only be found from eqn (1) by finding the ratio $\omega_1 : \omega_2 : \omega_3$ i.e by putting $I = I_1$ in (2) we obtain $\omega_1 : \omega_2 : \omega_3$ that yields the direction of $\vec{\omega}$ or PAI for $I = I_1$.

Find the M.I. & product of inertia of square lamina of side 'a' about x-y-z system when origin is taken at one vertex. Then set up the PAI & find out PMI.



For the square lamina $x_3 = 0$

$M = a^2 \sigma$

$$I_{11} = I_{22} = \int (y^2 + z^2)\, dm$$

$$= \iint x_2^2\, dx_1\, dx_2\, \sigma$$

$$= \sigma \int_0^a x_2^2\, dx_2 \int_0^a dx_1$$

$$= \frac{1}{3} M a^2$$

From $\perp^r$ axis theorem $I_{33} = I_{11} + I_{22} = \frac{2}{3} M a^2$

$$I_{21} = I_{12} = -\int x_1 x_2\, dx_1\, dx_2\, \sigma = -\sigma \int_0^a x_1\, dx_1 \int_0^a x_2\, dx_2 = -\frac{1}{4} M a^2$$

& as $x_3 = 0$ $\quad I_{13} = I_{31} = 0$, $I_{23} = I_{32} = 0$

$$I_{\hat{n}} = \begin{pmatrix} \frac{1}{3} M a^2 & -\frac{1}{4} M a^2 & 0 \\ -\frac{1}{4} M a^2 & \frac{1}{3} M a^2 & 0 \\ 0 & 0 & \frac{2}{3} M a^2 \end{pmatrix}$$

The characteristic determinant is

$$\begin{vmatrix} \frac{1}{3} M a^2 - I & -\frac{1}{4} M a^2 & 0 \\ -\frac{1}{4} M a^2 & \frac{1}{3} M a^2 - I & 0 \\ 0 & 0 & \frac{2}{3} M a^2 - I \end{vmatrix} = 0 \Rightarrow I_1 = \frac{1}{12} M a^2$$

$$I_2 = \frac{7}{12}$$

$$I_3 = \frac{2}{3} M a^2$$

To find PAI corresponding to $I = I_1 = \frac{1}{12} M a^2$

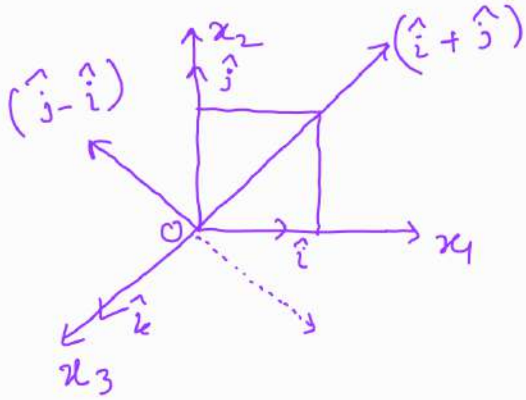
$$\begin{pmatrix} \frac{1}{3}Ma^2 - I_1 & -\frac{1}{4}Ma^2 & 0 \\ -\frac{1}{4}Ma^2 & \frac{1}{3}Ma^2 - I_1 & 0 \\ 0 & 0 & \frac{2}{3}Ma^2 - I_1 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = 0$$

$$\omega_1 = \omega_2 \, , \quad \omega_3 = 0 \quad \Rightarrow \quad \vec{\omega} = (\hat{i} + \hat{j})\, \omega_1$$

Similarly for $I = I_2 = \frac{7}{12} M a^2$ $\quad \omega_1 = -\omega_2 , \; \omega_3 = 0$

$$\vec{\omega} = \pm (\hat{j} - \hat{i})\, \omega_2$$

for $I = I_3 = \frac{2}{3} M a^2$, $\omega_1 = \omega_2 = 0$, $\omega_3 \neq 0$ $\vec{\omega} = \omega_3 \hat{k}$



31/12/21

Euler's eqn of motion for rigid body

Newtonian Method : If a rigid body rotates under the action of a torque $\vec{\tau}$ with one point fixed then the eqn of motion is,

$$\vec{\tau} = \left(\frac{d\vec{L}}{dt}\right)_s \rightarrow (1)$$

where $\vec{L}$ is the angular momentum & its time derivative refers to the space set axes, as the eqn of motion holds in inertial frame.

The body coordinate system is rotating with instantaneous angular velocity $\vec{\omega}$. The relation between time derivative in space set & body set system is

$$\left.\frac{d\vec{L}}{dt}\right|_s = \left.\frac{d\vec{L}}{dt}\right|_b + \vec{\omega}\times\vec{L} \rightarrow (2)$$

$$\Rightarrow \quad \vec{\tau} = \frac{d\vec{L}}{dt} + \vec{\omega}\times\vec{L} \rightarrow (3)$$

Choosing PAI for body set axes & denoting I_1, I_2, I_3 as PMI we have

$$\vec{L} = I_1\omega_1\hat{i} + I_2\omega_2\hat{j} + I_3\omega_3\hat{k} \rightarrow (4)$$

where ω_i are the components along PAI.

From eqn (4)

$$\frac{d\vec{L}}{dt} = I_1\dot{\omega}_1\hat{i} + I_2\dot{\omega}_2\hat{j} + I_3\dot{\omega}_3\hat{k} \rightarrow (5)$$

(3) & (5) $\Rightarrow$ $$\vec{\tau} = I_1\dot{\omega}_1\hat{i} + I_2\dot{\omega}_2\hat{j} + I_3\dot{\omega}_3\hat{k} + (\vec{\omega}\times\vec{L}) \rightarrow (5a)$$

In component form the above eqn is

$$\left.\begin{aligned} \tau_1 &= I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 \\ \tau_2 &= I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 \\ \tau_3 &= I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 \end{aligned}\right\} (6) \rightarrow \text{Euler's eqn}$$

Force free motion of symmetrical top:

When a rigid body is not subjected to any net torque, the Euler's eqn of motion of the body with one point fixed would be

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3, \quad I_2\dot{\omega}_2 = (I_3 - I_1)\omega_1\omega_3, \quad I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2$$

For a symmetrical top $I_1 = I_2$ & 3rd PAI is z axis coincides with axis of symmetry of the body. In this case Euler's eqn of motion will be

$$I_1\dot{\omega}_1 = (I_1 - I_3)\omega_2\omega_3 \rightarrow (1) \quad I_2\dot{\omega}_2 = (I_3 - I_1)\omega_1\omega_3 \rightarrow (2) \quad \dot{\omega}_3 = 0 \rightarrow (3)$$

(3) $\Rightarrow$ $\omega_3 = \text{const}$ Putting $\Omega = \frac{I_3 - I_1}{I_1}\omega_3 = \text{cst}$

(1) $\Rightarrow$ $\dot{\omega}_1 = -\Omega\omega_2$, $\dot{\omega}_2 = \Omega\omega_1 \Leftarrow (2)$

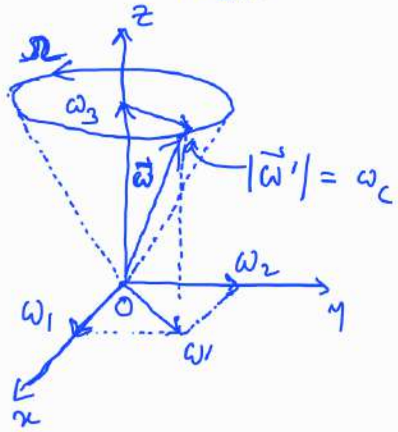
$\Rightarrow$ $\ddot{\omega}_1 = -\Omega\dot{\omega}_2 = -\Omega^2\omega_1 \Rightarrow \ddot{\omega}_1 + \Omega^2\omega_1 = 0$ & $\ddot{\omega}_2 + \Omega^2\omega_2 = 0$

$\Rightarrow$ $\omega_1 = \omega_c \sin(\Omega t)$, ω_c is some cst & we choose initial condition $\omega_1 = 0$ at $t = 0$ so that phase cst is zero.

Putting $\dot{\omega}_1$ we have $\omega_2 = \omega_c \sin \Omega t$

$$\vec{\omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k} = \vec{\omega}' + \omega_3 \hat{k}$$

$$\therefore \quad \omega_1^2 + \omega_2^2 = \omega_c^2 \rightarrow (3)$$



Eqn (3) is the eqn of circle having radius ω_c. If $\hat{i}, \hat{j}$ are unit vectors along two PAI other than symmetry axis then the vector

$$\vec{\omega}' = \omega_1 \hat{i} + \omega_2 \hat{j}$$

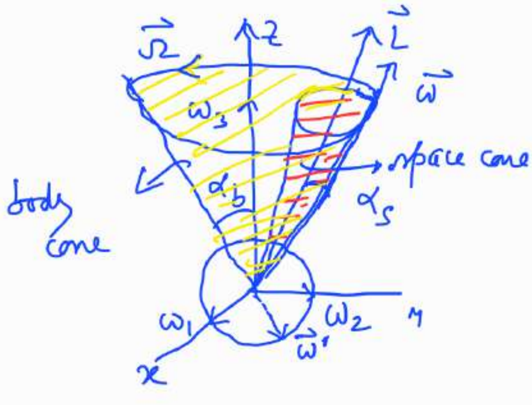
$= (\omega_c \cos \Omega t \, \hat{i} + \omega_c \sin \Omega t \, \hat{j})$ has a cnst. magnitude ω_c, rotating about third PA (symmetry axis) with an angular velocity $\Omega = \frac{I_3 - I_1}{I_1} \omega_3$. Total ang. velocity $\vec{\omega} = \vec{\omega}' + \omega_3 \hat{k}$, $|\vec{\omega}| = \sqrt{\omega_c^2 + \omega_3^2} =$ cnst. Thus the angular vel $\vec{\omega}$ having cnst magnitude precesses about z axis with cnst frequency Ω. We observe that $\vec{\omega}$ moves on the surface of a cone about the axis of symmetry with cnst ang. vel. Ω. This precessional motion is w.r.t. principal (body) axis fixed in the body which are themselves rotating in space with the ang. vel $\vec{\omega}$ relative to $\vec{\Omega}$.

Frequency & period of precession of ang. vel. $\vec{\omega}$ (or the axis of rotation) about the axis of symmetry (z axis) are given by

$$f = \frac{\Omega}{2\pi} = \frac{\omega_3}{2\pi} \frac{I_3 - I_1}{I_1} \qquad T = \frac{2\pi}{\omega_3} \frac{I_1}{I_3 - I_1}$$

If the angle between symmetry axis & instantaneous axis ($\vec{\omega}$) is α_b then

$$\tan \alpha_b = \frac{|\vec{\omega}'|}{|\vec{\omega}_3|} = \frac{\omega_c}{\omega_3} = \text{cnst}$$



The instantaneous axis rotates about symmetry axis (body axis) in the form of a cone with semi vertical angle α_b, called body cone.

In order to fix the motion in space we must locate $\vec{\omega}$ w.r.t. a direction fixed in space. Since $\vec{\tau} = 0$, $\vec{L}$ has fixed direction & magnitude in space.

$$\cos\alpha_s = \frac{\vec{\omega}\cdot\vec{L}}{\omega L} = \frac{2T}{\omega L} = \text{const} \qquad \alpha_s = \angle \vec{L}, \vec{\omega}$$

Thus instantaneous axis $\vec{\omega}$ makes a const. angle α_s with the $\vec{L}$ i.e moves on the surface of a cone with the direction of $\vec{L}$ as axis and apex at the centre of mass. This is called space cone.

The instantaneous axis of rotation is the line of contact between space cone & body cone at any instant. This axis in the body being instantaneously at rest, the body cone rolls without slipping around space cone.