1. Fundamental of Statistical and Thermal Physics - Federick Reif

2. Stat Mech - Pathria

3. Stat. Mech. - B. B. Laud

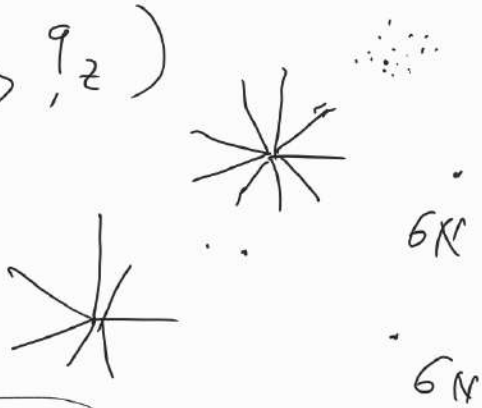
4. Stat. Mech - Suresh Chandra + Mohit Kr. Sharma

What is Stat. Mech?

(N)

$(3p, 3q)$ $(p_x, p_y, p_z, q_x, q_y, q_z)$

$\{p_i, q_i ; i = 1 - \overset{6N}{N}\}$



$\boxed{\text{phase point / representation}}$ (N, V, E)

$$E = \sum_i \frac{p_i^2}{2m_i}$$

08/04/21

1. Special Th. of Relativity
   – Robert . Resnick
2. Special Relativity. S. P. Puri
3. Do → P. L. Sardesai

Velocity of light w.r.t. lab

For the path → $M \to M_1$

$$\vec{v}_{light, lab} = \vec{v}_{light, ether} + \vec{v}_{ether, lab}$$

$$V = c - v$$

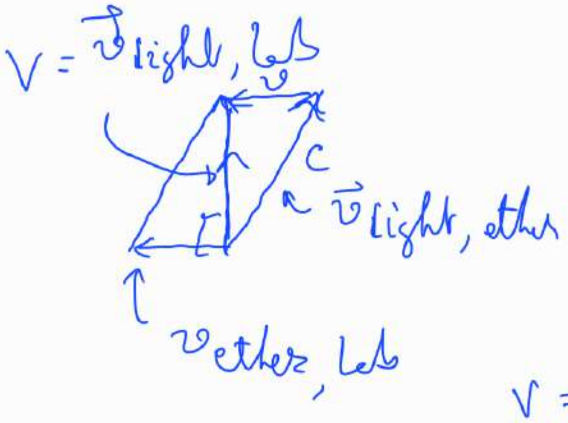
time taken for $M \to M_1$ $\quad t_1' = \dfrac{L_1}{c-v}$

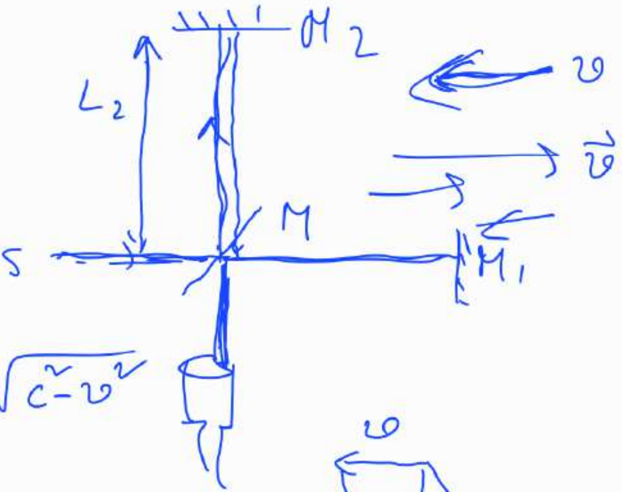
$$V' = c + v$$

$$t_2' = \frac{L_1}{c+v}$$

$$t_1 = \frac{L_1}{c-v} + \frac{L_1}{c+v} = \frac{2L_1}{c} \frac{1}{1 - v^2/c^2}$$

$\underline{M \to M_2}$:





$$V = \sqrt{c^2-v^2}$$

$$t_1'' = \frac{2\,2}{\sqrt{c^2-v^2}}$$

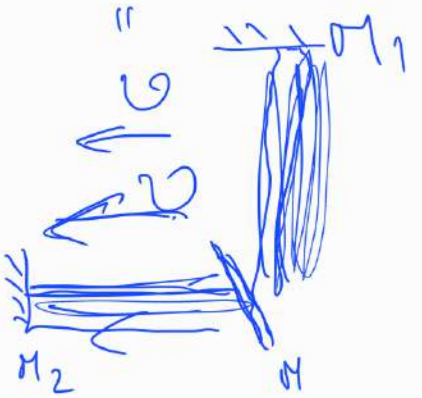
$$t_2 = \frac{2L_2}{c}\,\frac{1}{\sqrt{1-v^2/c^2}}$$

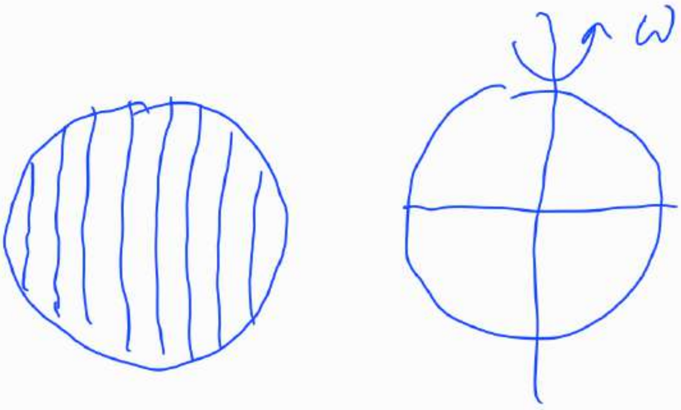
$$\Delta t = t_2 - t_1 = \left(\frac{2L_2}{c}\,\frac{1}{\sqrt{\;}} - \frac{2L_1}{c}\,\frac{1}{1-v^2/c^2}\right)$$

Phase change due to this δt

$$\Delta\varphi = \frac{\Delta t}{T}2\pi = \frac{\Delta t}{\lambda}(2\pi c)$$

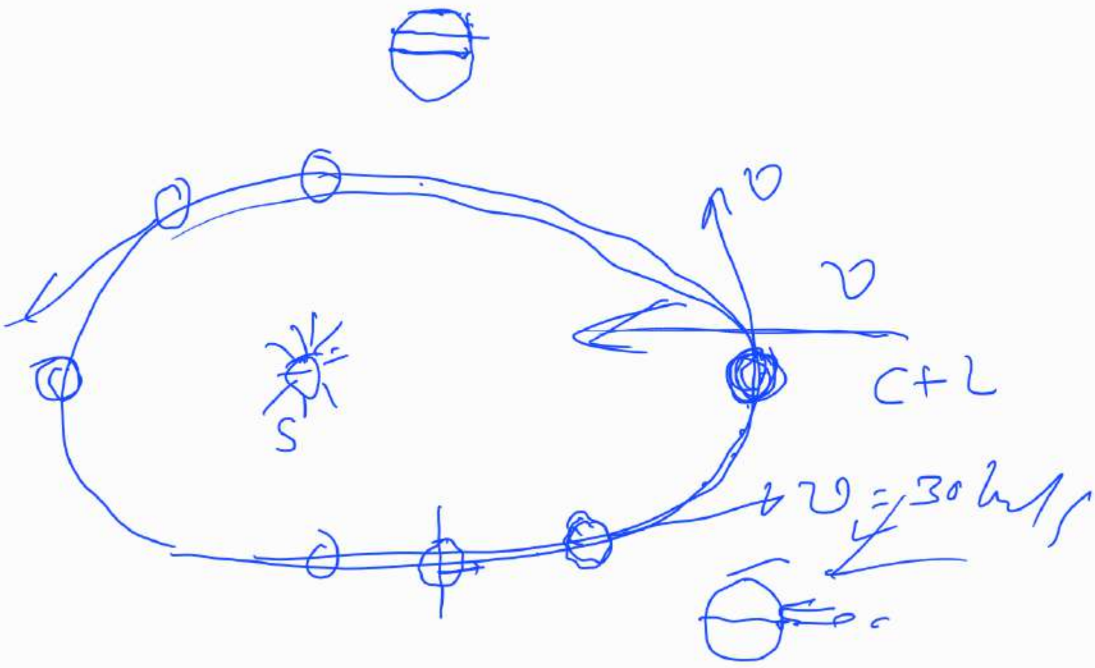
30 km/s





$$\Delta t' = \frac{2L_2}{c}\,\frac{1}{\sqrt{1-v^2/c^2}} - \frac{2L_1}{c}\,\frac{1}{\sqrt{1-v^2/c^2}}$$

$$\Delta\varphi' = \frac{2\pi c}{\lambda}\,\Delta t'$$



$$n = \frac{\Delta\varphi' - \Delta\varphi}{2\pi} = \frac{L_1+L_2}{\lambda}\,\frac{v^2}{c^2}$$

$\lambda = 5.5\times10^{-7}\,m \quad \frac{v}{c} = 10^{-4} \quad n =$

$L_1+L_2 = 22\,m$

$n = 0.4$

$= 0$ ✓

L

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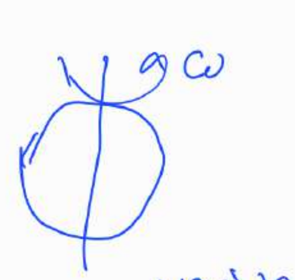
Generalised ang. momentum & Spin

$\vec{L} \neq \vec{r} \times \vec{p}$, $\quad \vec{S} = I\vec{\omega}$

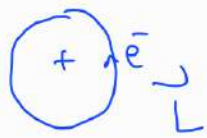
$\Uparrow$



$\vec{J} = \vec{L} + \vec{S}$

$\Downarrow \quad \Downarrow \quad \Downarrow$ no motion

Generalised ag. mom



1921

$\Downarrow$

Stern-Gerlach expt

$\Downarrow$

established evidence of spin-ang. mom

Anomalous Zeeman effect (1896)

~~exist~~ forces us to consider spin of electron.

has nothing with spin motion like orbital motion.

1st proposed by Uhlenbeck/Goudsmith

Theory of e^- spin → Pauli (1924)
→ Dirac (1928)

Theory of spin :

$\vec{L} \xrightarrow[Q]{} \hat{L} \Rightarrow \quad [\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z$

$\vec{S} \xrightarrow[Q]{} \hat{S} \qquad [\hat{S}_y, \hat{S}_z] = i\hbar \hat{S}_x$

$$[\hat{S}_i, \hat{S}_j] = i\hbar \epsilon_{ijk} \hat{S}_k$$

$$\epsilon_{ijk} = \pm 1$$

$$\hat{S}^2 |s, m_s\rangle = \hbar^2 s(s+1) |s, m_s\rangle, \qquad [\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$$

$$\hat{S}_z |s, m_s\rangle = \hbar m_s |s, m_s\rangle$$

$$\hat{S}_\pm |s, m_s\rangle = \hbar [s(s+1) - m_s(m_s \pm 1)]^{1/2} |s, (m_s \pm 1)\rangle$$

$$\hat{S}_\pm = \hat{S}_x \pm i \hat{S}_y$$

↳ $\hat{L}_z \to Y_{lm}(\theta, \varphi)$

$\hat{S}, S_s$ → no eigenfunction depending on (r, θ, φ)

Matrix form of spin operator

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_{+1} + b\chi_{-1} \qquad \pm\frac{1}{2}\hbar$$

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle = |\uparrow\uparrow\rangle$$

$$\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle = |\uparrow\downarrow\rangle$$

$$\hat{S}^2 \rightarrow \quad \hat{S}^2\chi_+ = \frac{3}{4}\hbar^2\chi_+ \quad , \hat{S}^2\chi_- = \frac{3}{4}\hbar^2\chi_-$$

$$\hat{S}^2 \rightarrow \begin{pmatrix} d & g \\ f & k \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{3}{4}\hbar^2\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} d \\ f \end{pmatrix} = \begin{pmatrix} \frac{3}{4}\hbar^2 \\ 0 \end{pmatrix}$$

$d = \frac{3}{4}\hbar^2$, $f = 0$ similarly $\begin{pmatrix} g \\ k \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{3}{4}\hbar^2 \end{pmatrix}$

$$\hat{S}^2 = \frac{3}{4}\hbar\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{S}_z\chi_+ = \frac{\hbar}{2}\chi_+ \qquad \hat{S}_z\chi_- = -\frac{\hbar}{2}\chi_-$$

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$$

$$\hat{S}_+\chi_- = \hbar\chi_+ \qquad \hat{S}_-\chi_+ = \hbar\chi_+$$

$$\hat{S}_+\chi_+ = 0 \qquad \hat{S}_-\chi = 0$$

$$\hat{S}_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \hat{S}_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\hat{S}_x = \frac{1}{2}(\hat{S}_+ + \hat{S}_-) \qquad \hat{S}_y = \frac{1}{2i}(\hat{S}_+ - \hat{S}_-)$$
$$= \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Pauli Spin matrices

$$\hat{S} = \frac{\hbar}{2}\hat{\sigma}$$

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\hat{S}^2, \hat{S}_i \rightarrow$ Hermitian

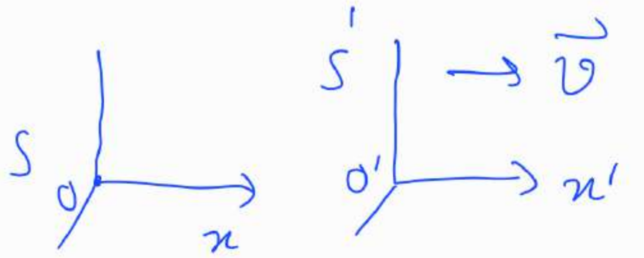
<u>Electron in a magnetic field</u>

<u>G.T. eqn. not suitable to</u>

<u>express Einstein postulate</u>

(2nd) Velocity of light is same in vacuum in all inertial frames

(1st) Laws of physics are same in all inertial frames



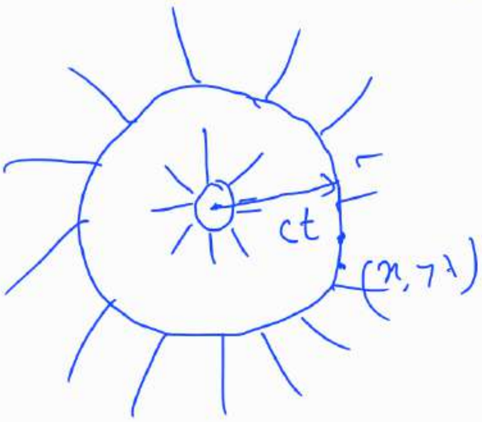
$\cdot P \; (x, y, z, t)$
(x', y', z', t')

$x' = x - vt$

$$\begin{cases} y' = y \\ z' = z \\ t' = t \end{cases}$$

$x = x' + vt'$

$t = t' = 0, \quad O, O'$ coincident



$x^2 + y^2 + z^2 - c^2 t^2 = 0$

S frame

$(x' + vt')^2 + y'^2 + z'^2 - c^2 t'^2 = 0$

S' frame

$x'^2 + y'^2 + z'^2 - c^2 t'^2 = 0$

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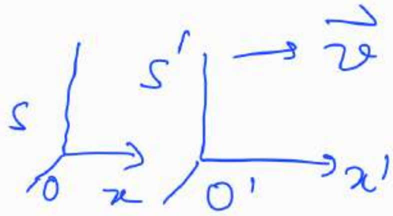
G. T. not suitable to describe postulate of relativity

Lorentz Transformation eqn :



$$\begin{cases} x' = \gamma (x - vt) \\ t' = \gamma \left(t - \frac{\beta x}{c}\right) \\ y' = y \\ z' = z \end{cases}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = \frac{|\vec{v}|}{c}$$

$\cdot P(x, y, z, t)$

(x', y', z', t')

$$\left.\begin{aligned} x' &= a_{11}x + a_{12}y + a_{13}z + a_{14}t \\ y' &= a_{21}x + a_{22}y + a_{23}z + a_{24}t \\ z' &= a_{31}x + a_{32}y + a_{33}z + a_{34}t \\ t' &= a_{41}x + a_{42}y + a_{43}z + a_{44}t \end{aligned}\right\} \to (1)$$

Homogeneity of space / Isotropy of space

L = 1m

$x_1 = 2m \quad x_2 = 3m$

$x_2 - x_1 = 1m$

$x_1 = 8 \quad x_2 = 9$

$x_2 - x_1 = 1$

$$x' = a_{11} x^2$$

$$x'_2 - x'_1 = 5 a_{11} \quad a_{11}(3^2 - 2^2)$$

$$= a_{11}(9^2 - 8^2) = 17 a_{11}$$

$$5 a_{11}$$

x-axis coincides with x', $\forall t$

$\forall \quad y = 0,\ z = 0$ (x-axis) it always

$y' = 0,\ z' = 0$ (x'-axis)

$$\left. \begin{aligned} y' &= a_{22} y + a_{23} z \\ z' &= a_{32} y + a_{33} z \end{aligned} \right\}$$

x-y plane ($z = 0$) should transform over x'-y' plane ($z' = 0$) & x-z plane ($y = 0$) implies x'-z' plane ($y' = 0$)

$$\left. \begin{aligned} y' &= a_{22} y \\ z' &= a_{33} z \end{aligned} \right\}$$

S y | L = unit length

S' | $y' = a_{22} \times 1 = a_{22}$

S $\qquad\qquad S'$

$L = 1$

$$y = \frac{y'}{a_{22}} = \frac{1}{a_{22}} \, y$$

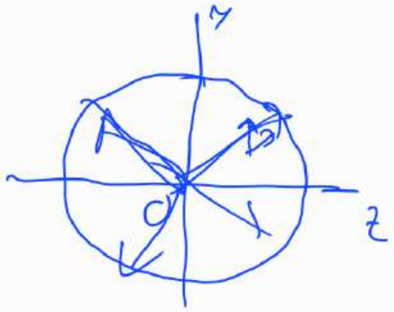
$$a_{22} = \frac{1}{a_{22}} \Rightarrow a_{22} = +1$$

$$a_{33} = 1 \qquad y = y'$$

$$z = z'$$

$$t' = a_{41} x + a_{42} y + a_{43} z + a_{44} t$$

$$= a_{41} x + a_{44} t$$



$$x' = a_{11} x + a_{12} y + a_{13} z + a_{14} t$$

$x' = 0$ must be identical to $x = vt$

$$x' = a_{11}(x - vt)$$

$$= a_{11} x - a_{11} v t = a_{11} x + a_{14} t$$

$$a_{14} = -v a_{11}$$

$$x^2+y^2+z^2-c^2t^2$$

$$x'^2+y'^2+z'^2-c^2t'^2$$

$$a_{11}^2(x-vt)^2+y^2+z^2=c^2(a_{41}x+a_{44}t)^2$$

$x, t \to$

$$c^2a_{44}^2-a_{11}^2v^2=c^2 \to (1)$$

$$a_{11}^2-c^2a_{41}^2=1 \to (2)$$

$$c^2a_{41}a_{44}+a_{11}^2v=0$$

$$a_{44}=a_{11} \qquad a_{41}=a_{44}\frac{-v}{c^2}$$

$$a_{11}=\frac{1}{\sqrt{1-v^2/c^2}}$$

$$\begin{cases} x'=\gamma(x-vt) \\ t'=\gamma\left(t-\frac{vx}{c^2}\right) \\ y'=y \\ z'=z \end{cases}$$

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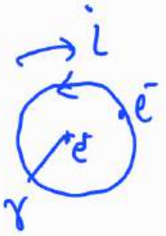
Electron in a magnetic field

Larmor frequency



$$\mu = iA = -\frac{e}{T}\pi r^2$$

$$l = mvr = m\frac{2\pi r}{T}r$$

$$\Rightarrow \mu = -\frac{e}{2m}l \equiv \gamma l$$

$$\gamma = -\frac{e}{2m}$$

$\Downarrow$

gyromagnetic ratio

S

$$\vec{\mu}_S \equiv \gamma \vec{S}$$

$$H = -\vec{\mu}\cdot\vec{B} = -\gamma\vec{B}\cdot\vec{S}$$

$$= -\gamma B_0 S_z$$

$$= -\frac{\gamma B_0 \hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$\uparrow \vec{B}\ \hat{k}$

$\vec{B} = B_0\hat{k}$

χ_+ , $E_+ = -(\gamma B_0 \hbar)/2$

χ_- $\quad E_- = (\gamma B_0 \hbar)/2$

$\vec{\mu} \parallel \vec{B}$

$$i\hbar \frac{\partial \chi}{\partial t} = H\chi$$

$$\chi(t) = a\chi_+ e^{-iE_+ t/\hbar} + b\chi_- e^{-iE_- t/\hbar}$$

$$= \begin{pmatrix} a e^{i\gamma B_0 t/2} \\ b e^{-i\gamma B_0 t/2} \end{pmatrix}$$

$$\chi(0) = \begin{pmatrix} a \\ b \end{pmatrix} \qquad |a|^2 + |b|^2 = 1$$

$$a = \cos \alpha/2 \qquad b = \sin \alpha/2$$

$$\chi(t) = \begin{pmatrix} \cos\alpha/2 \; e^{i\gamma B_0 t/2} \\ \sin\alpha/2 \; e^{-i\gamma B_0 t/2} \end{pmatrix}$$

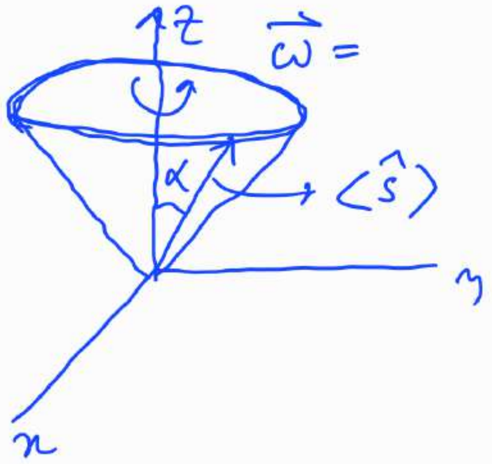
$$\langle \hat{S}_x \rangle = \chi(t)^\dagger \hat{S}_x \chi(t)$$

$$= \qquad \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\langle \hat{S}_x \rangle = \frac{\hbar}{2} \sin\alpha \cos(\gamma B_0 t)$$

$$\langle \hat{S}_y \rangle = \chi(t)^\dagger \hat{S}_y \chi(t) = -\frac{\hbar}{2} \sin\alpha \sin(\gamma B_0 t)$$

$$\langle \hat{S}_z \rangle = \chi(t)^\dagger \hat{S}_z \chi(t) = \frac{\hbar}{2} \cos\alpha$$



$|\vec{\omega}|$ = Larmor frequency $= \gamma B_0$

Stern-Gerlach expt (1921)

$$\vec{F} = \nabla(\vec{\mu} \cdot \vec{B})$$

$$\Downarrow$$

$$|\vec{S}|$$

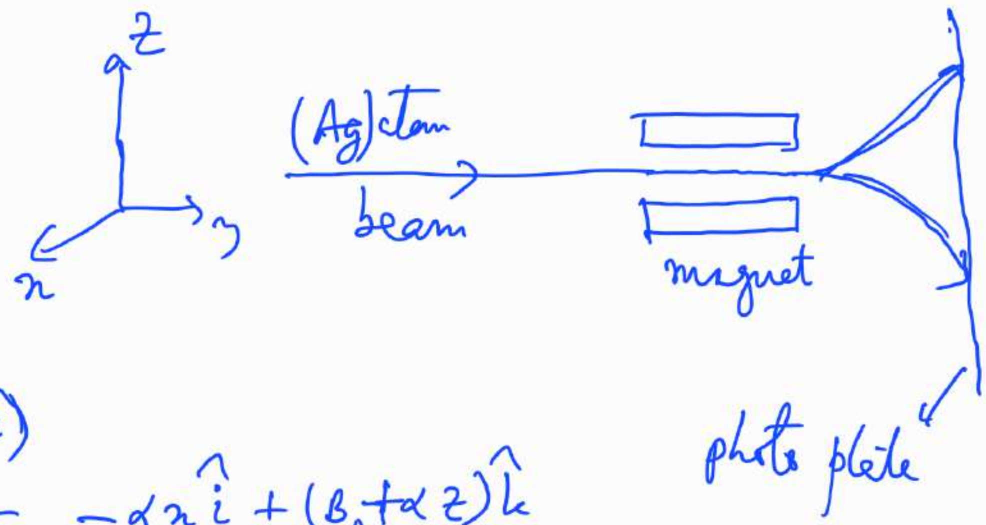
P.E. $= U = -\vec{\mu} \cdot \vec{B}$

$\vec{F} = -\vec{\nabla} U$

$\vec{B}$

$-e^-$

L.F. $= \vec{F} = q(\vec{v} \times \vec{B})$



$$\vec{B}(x,y,z) = -\alpha x \hat{i} + (B_0 + \alpha z)\hat{k}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{F} = \gamma \alpha (-S_x \hat{i} + S_z \hat{k})$$

$$F_z = \gamma \alpha S_z$$

Quantum Mechanical Treatment

$$H(t) = 0 \quad t < 0$$

$$-\gamma (B_0 + \alpha z) S_z \quad 0 \le t \le T$$

$$= 0 \quad t > T$$

$$\chi(t) = a\chi_+ + b\chi_-, \quad t \le 0$$

$$\chi(t) = a\chi_+ e^{-iE_+ t/\hbar} + b\chi_- e^{-iE_- t/\hbar}$$

$$0 \le t \le T$$

$$E_\pm = \mp \gamma (B_0 \alpha z) \frac{\hbar}{2} \quad [H = -\gamma \vec{B} \cdot \vec{S}]$$

$$x(t) = \underline{\left(a e^{i\gamma T B_0/2} x_+\right) e^{i(\alpha\gamma T/2)z}}$$

$$+ \underline{\left(b e^{-i\gamma T B_0/2} x_-\right) e^{-i(\gamma T B_0))z}}$$

$$t \geq T$$

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$$

spin-up comp has momentum $p_z = \frac{\alpha\gamma T \hbar}{2}$

$$_{47}Ag = 1s^2 2s^2 2p^6 3s^2 \cdots 4p^6 5s^1 4d^{10}$$

$L = 0, \; S = 0$

$5s^1 \;\; L = 0, \;\; s = \frac{1}{2}$

No. of Traces $(2J+1) = 2$

Addition of ang. momenta

$j_1, j_2, j_3 \cdots$

$j = j_1 + j_2, \; j_1 + j_2 - 1, \cdots |j_1 - j_2|$

$m = (2j+1), \; m = j, j-1, j-2 \cdots, -j$

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spectra of H-atom & Fine structure

$$E_n^1 = \langle \psi_n^0 | H' | \psi_n^0 \rangle$$

$|\psi_n^0\rangle$ = complete set orthonormal eigenfunction E_n^0

$$H = \left(-\frac{\hbar^2}{2m}\nabla^2 - \frac{e^2}{4\pi\epsilon_0 r} \right) \longrightarrow (1)$$

$$E_n = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2} = \frac{E_1}{n^2}$$

$n = 1, 2, 3$

$E_1 \to$ ground state energy
$= -13.6$ eV

$$= -\frac{1}{2} mc^2 \frac{\alpha^2}{n^2} \qquad \alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} = \frac{1}{137}$$

fine structure const

$$T = \frac{p^2}{2m} = -\frac{\hbar^2}{2m}\nabla^2 \qquad p \to \frac{\hbar}{i}\nabla$$

$$\text{K.E. } T = \frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2, \quad p = \frac{mv}{\sqrt{1-v^2/c^2}}$$

$$E^2 = \left[\frac{mc^2}{\sqrt{1-v^2/c^2}}\right]^2 = p^2c^2 + m^2c^4$$

$$\Rightarrow T = (p^2c^2 + m^2c^4)^{1/2} - mc^2$$

$$H'_r = -\frac{p^4}{8m^3c^2}$$

$$E^1_r = \langle H'_r \rangle = -\frac{1}{8m^3c^2}\underbrace{\langle \psi | p^4 | \psi \rangle}_{p^2p^2}$$

$$=$$

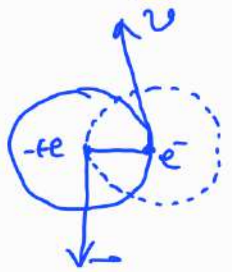
$$p^2\psi = 2m(E-V)\psi$$

$$E^1_r = -\frac{1}{2mc^2}\left(E^2 - 2E\langle V\rangle + \langle V^2\rangle\right)$$

$$V(r) = -\frac{e^2}{4\pi\epsilon_0 r} \qquad \left\langle \frac{1}{r}\right\rangle = \frac{1}{n^2a_0} \qquad \left\langle \frac{1}{r^2}\right\rangle = \frac{1}{(l+\frac{1}{2})n^3a_0^2}$$

$$E^1_n = -\frac{1}{2mc^2}\left[E_n^2 + 2E_n\left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{1}{n^2a_0} + \frac{e^2}{4\pi\epsilon_0}\frac{1}{(l+\frac{1}{2})n^3a_0^2}\right]$$

$$= -\frac{E_n^2}{2mc^2}\left[\frac{4n}{l+\frac{1}{2}} - 3\right], \quad E_n = -\frac{e^2}{8\pi\epsilon_0 a_0 n^2}$$



$-\vec{v} \rightarrow$ vel of proton in electron's frame

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{\vec{J}\times\vec{r}}{r^3} = -\frac{\mu_0 e}{4\pi}\frac{\vec{v}\times\vec{r}}{r^3} \qquad J = e(-\vec{v})$$

$$= \frac{e\mu_0}{4\pi m}\frac{\vec{L}}{r^3} \qquad \vec{L} = \vec{r}\times m\vec{v} = -m\vec{v}\times\vec{r}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{e}{mc^2}\frac{\vec{L}}{r^3} \qquad c = \frac{1}{\sqrt{\epsilon_0\mu_0}}$$

$$\mu_e = \underset{\Downarrow}{2}\left(-\frac{e}{2m}\vec{S}\right) = -\frac{e}{m}\vec{S}$$

$$H'_{SO} = -\vec{\mu}_e\cdot\vec{B} = \frac{e^2}{4\pi\epsilon_0}\frac{1}{m^2c^2r^3}\vec{S}\cdot\vec{L}$$

$$= \underline{\frac{e^2}{8\pi\epsilon_0}\frac{1}{m^2c^2r^3}}\,\vec{S}\cdot\vec{L}$$

$$\vec{J}^2 = (\vec{L}+\vec{S})\cdot(\vec{L}+\vec{S}) \Rightarrow \vec{S}\cdot\vec{L} = \frac{1}{2}\left(J^2 - L^2 - S^2\right)$$

$$\vec{L}\cdot\vec{S} = \frac{\hbar^2}{2}\left[j(j+1) - l(l+1) - s(s+1)\right]$$

$$E^1_{SO} = \frac{E_n^2}{mc^2}\left\{\frac{n\left[j(j+1)-l(l+1)-3/4\right]}{l(l+\frac{1}{2})(l+1)}\right\}$$

$$\rightarrow \frac{0}{0} \text{ for } l=0$$

$j = l+\frac{1}{2}$ $\quad$ $j = l-\frac{1}{2}$ is not applicable for $l=0$ case

$$E^1_{SO}\Big|_{j=l+\frac{1}{2}} = \frac{(E_n)^2}{mc^2}\frac{n}{(l+\frac{1}{2})(l+1)}$$

$$E = E_r^1 + E_{so}^1 = \frac{E_n^2}{2mc^2}\left[3 + 2n\left\{\frac{j(j+1) - 3l(l+1) - \frac{3}{4}}{l(l+\frac{1}{2})(l+1)}\right\}\right]$$

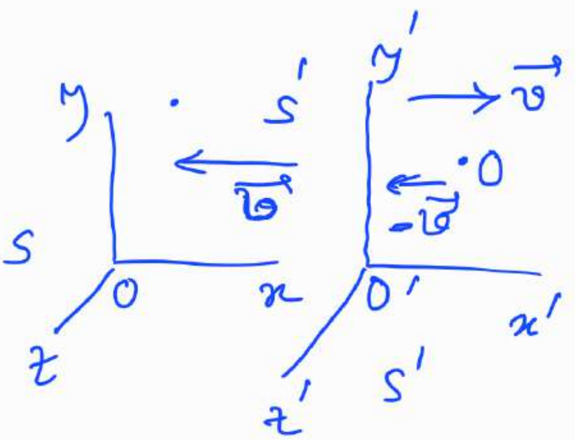
$$E_{n,j}^T = -\frac{13.6}{n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)\right]$$

L. T. eqn :

$S \to S'$

$$x' = \gamma(x - vt) \qquad y' = y$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \qquad z' = z$$

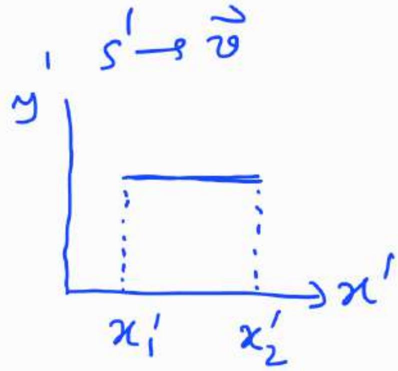


Inverse L.T.

• $P(x, y, z, t)$
(x', y', z', t')

$$\left.\begin{aligned} x &= \gamma(x' + vt') \\ t &= \gamma\left(t' + \frac{vx'}{c^2}\right) \end{aligned}\right\} S' \to S$$

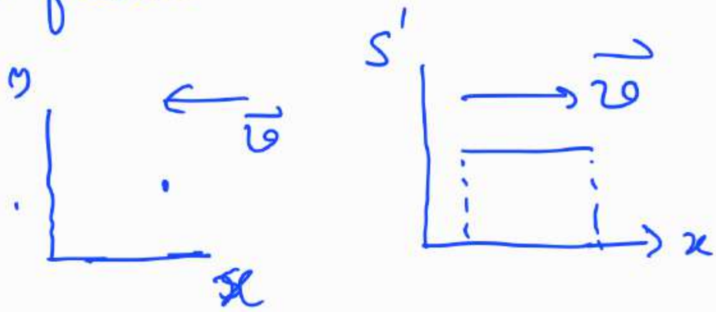
1. **Length Contraction**



Rest length of rod in
S' frame $= L_0 = (x_2' - x_1')$

$\to$ proper length

What would be the length of the rod in S-frame



End points coordinate of rod from S-frame x_2, x_1

$$x_2' = \gamma(x_2 - vt_2)$$
$$x_1' = \gamma(x_1 - vt_1)$$

for measurement in S-fram

$$t_2 = t_1$$

$$L_0 = x_2' - x_1' = \gamma(x_2 - x_1) \quad (t_2 = t_1)$$
$$= \gamma L = \frac{L}{\sqrt{1-\beta^2}} \quad \beta = \frac{v}{c}$$

$$L = L_0\left(\sqrt{1-\beta^2}\right) \rightarrow < 1$$

↳ length observed from S where w.r.t. which rod is moving

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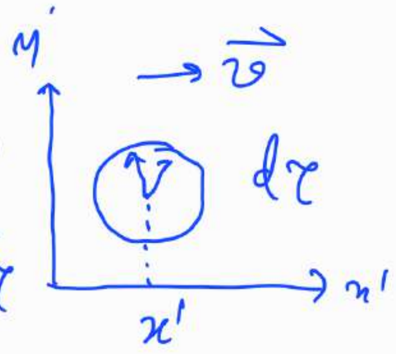
Time dilation : Moving clock runs slow

S'-frame event-1



$E_1 : x', t_1'$ $\quad S \rightarrow S'$

$\Delta t > d\tau$

$E_2 : x', t_2'$

Time interval between events E_2, E_1

$d\tau = t_2' - t_1' \rightarrow$ proper time

events are happening at same position, $\Rightarrow$ clock at rest w.r.t. S'-frame

S-frame :

Time of occurrence of event E_1 from S-frame

$$t_1 = \gamma\left(t_1' + \frac{v x'}{c^2}\right) \qquad t_2 = \gamma\left(t_2' + \frac{v x'}{c^2}\right)$$

$$\Delta t = t_2 - t_1 = \gamma\left(t_2' - t_1'\right) = \gamma\, d\tau$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} > 1$$

$$\Delta t > d\tau$$

Time interval measured from S-frame

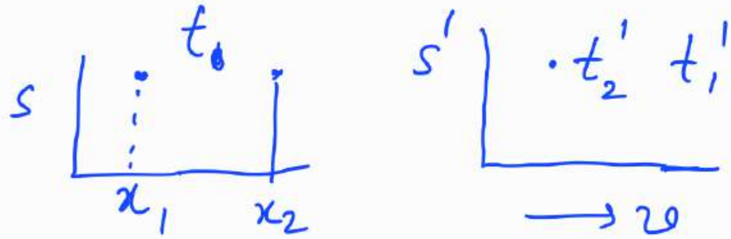
> that measured in S'-frame

where clock is at rest

$$L = L_0\sqrt{1-\beta^2} \rightarrow L < L_0$$

$$\delta t = \frac{d\tau}{\sqrt{1-\beta^2}} \rightarrow \delta t > d\tau$$

<u>Relativity of simultaneity:</u>



$E_1\ (x, t)$

$E_2\ (x_2, t)$

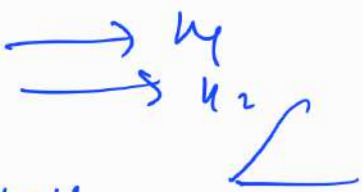
$$E_1,\ t_1' = \frac{t - vx_1/c^2}{\sqrt{1-\beta^2}}$$

$$E_2,\ t_2' = \frac{t - vx_2/c^2}{\sqrt{1-\beta^2}}$$

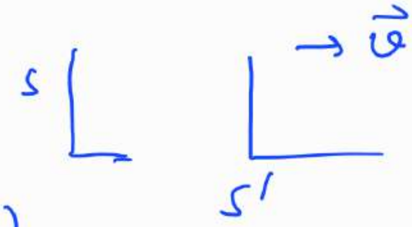
$$t_2' \neq t_1'$$

<u>Velocity addition Theorem:</u>

$\vec{u}_1,\ \vec{u}_2$



$$\vec{U} = \vec{u}_1 + \vec{u}_2 = u_1 \pm u_2$$



$\cdot (x, y, z, t)$

(x', y', z', t')

Velocity comp. of a body (P)

w.r.t. S-frame

$$u_x = \frac{dx}{dt} \quad u_y = \frac{dy}{dt}, \quad u_z = \frac{dz}{dt}$$

In S'-frame $\quad u'_x = \frac{dx'}{dt'} \quad u'_y = \frac{dy'}{dt'}, \quad u'_z = \frac{dz'}{dt'}$

$u_x \sim u'_x$? $\quad u_y \sim u'_y$?

$$x = \gamma(x' + vt')$$

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

$$u_x = \frac{dx}{dt} = \frac{\gamma(dx' + v\,dt')}{\gamma\left(dt' + \frac{v\,dx'}{c^2}\right)}$$

$$u_x = \frac{u'_x + v}{1 + \beta u'_x/c}$$

$$u_y = \frac{dy'}{\gamma(dt' + \beta dx')} = \frac{u'_y}{\gamma(1 + \beta u'_x/c)}$$

$$u_z = \frac{u'_z}{\gamma(1 + \beta u'_x/c)}$$

$$u'_x = \frac{dx'}{dt'} = \frac{\gamma(dx - v\,dt)}{\gamma\left(dt - \frac{v\,dx}{c^2}\right)} = \frac{u_x - v}{1 - \frac{v u_x}{c^2}}$$

$$= \frac{u_x - v}{(1 - \beta u_x/c)}$$

$$u' = \frac{u - v}{1 - \beta u/c} \Rightarrow u = \frac{u' + v}{1 + \beta u'/c}$$

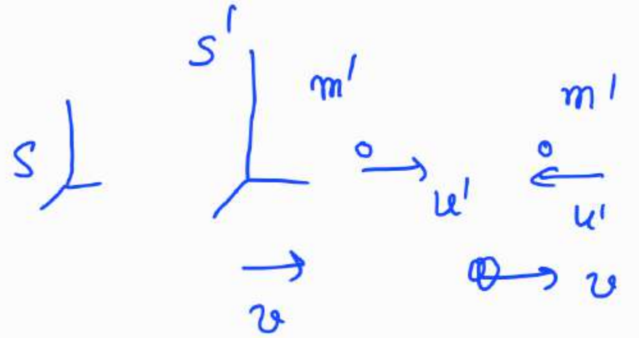
$$= (u' + v)$$

$$m = \frac{m_0}{\sqrt{1 - u^2/c^2}} \qquad \rho = \frac{m_0}{V = dx\,dy\,dz}$$

$\rho' = \frac{m}{V'}$ $\qquad$ $\rho_0 = \frac{m_0}{V}$

$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$ $\qquad$ $m_0 \to$ mass at rest

$m \to$ mass measured ~~from~~ ... moving with vel 'v'



$m'u' - m'u' = 0 \to S'$ frame

m_1, m_2

$u_1 \; u_2$

$m_1 u_1 + m_2 u_2 = (m_1 + m_2) v$

$\to$ S-frame

$$u_1 = \frac{u' + v}{1 + \frac{u'v}{c^2}}, \quad u_2 = \frac{-u' + v}{1 - \frac{u'v}{c^2}}$$

$$m_1 u_1 + m_2 u_2 = (m_1 + m_2) v$$

$$\Rightarrow \frac{m_1}{1 + \frac{u'v}{c^2}} \left[u' + v - v - \frac{u'v^2}{c^2} \right] = \frac{m_2}{1 - \frac{u'v}{c^2}} \left[u' - v + v - \frac{u'v^2}{c^2} \right]$$

$$\Rightarrow \frac{m_1}{m_2} = \frac{1 + \frac{u'v}{c^2}}{1 - \frac{u'v}{c^2}}$$

$$\left(1-\frac{u_1^2}{c^2}\right) = \frac{\left(1-\frac{u'^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{\left(1+\frac{u'v}{c^2}\right)^2}$$

$$\left(1-\frac{u_2^2}{c^2}\right) = \frac{\left(1-\frac{u'^2}{c^2}\right)\left(1-v^2/c^2\right)}{\left(1-\frac{u'v}{c^2}\right)^2}$$

$$\frac{m_1}{m_2} = \frac{1+\frac{u'v}{c^2}}{1-\frac{u'v}{c^2}} = \frac{\left(1-\frac{u_2^2}{c^2}\right)^{1/2}}{\left(1-\frac{u_1^2}{c^2}\right)^{1/2}}$$

$m_0 \rightarrow$ mass under this condition when are at rest

$$m_1\sqrt{1-\frac{u_1^2}{c^2}} = m_2\sqrt{1-\frac{u_2^2}{c^2}} = m_0$$

$$\Rightarrow \boxed{m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}}$$

Mass-energy equivalence: $E = mc^2 = \dfrac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}}$

$m_0 \rightarrow$ rest mass of a body

at $t=0$, $x=0$, $u=0$ $\quad \overrightarrow{F} \longrightarrow x$

$$\text{K.E.} \quad T = \int_0^x F\,dx = \int_0^u \frac{dp}{dt}\,dx = \int_0^u u\,dp$$

$$= \int_0^u u\,d(mu) = \int_0^u (mu\,du + u^2 dm)$$

$$m = \frac{m_0}{\sqrt{1-u^2/c^2}} \Rightarrow mu\,du + u^2 dm = c^2 dm$$

$$\Rightarrow T = \int_{m_0}^{m} c^2 dm = c^2(m-m_0) \Rightarrow mc^2 = T + m_0 c^2$$

We take $E(u) = mc^2$ $\qquad E(0) = m_0 c^2$

value of $E(u)$ for K.E. $= 0$ at $t = 0$

$$E(u) = E(0) + T(u)$$

$E(0)$: rest energy $= m_0 c^2$

$$E = mc^2 = \frac{m_0}{\sqrt{1-u^2/c^2}} c^2$$

$$E^2 = p^2c^2 + m_0^2 c^4$$

$$p = mv = \frac{m_0 v}{\sqrt{1-v^2/c^2}} \Rightarrow p^2c^2 + m_0^2 c^4 = \frac{m_0^2 v^2 c^2}{1-v^2/c^2} + m_0^2 c^4$$

$$= m^2 c^4 = E^2$$

$$\Rightarrow \boxed{E^2 = c^2 p^2 + m_0^2 c^4}$$

$$\cancel{2E\frac{dE}{dt} = 2c^2 p \frac{dp}{dt} + 0}$$

$$\cancel{\Rightarrow E\frac{dE}{dt} = c^2 p F}$$

$$E^2 = c^2 p^2 + m_0^2 c^4 = c^2 \vec{p}\cdot\vec{p} + m_0^2 c^4$$

$$E\frac{dE}{dt} = c^2 \vec{p}\cdot\frac{d\vec{p}}{dt} + 0$$

$$= c^2 \vec{p}\cdot\vec{F}$$

$$m\frac{dE}{dt} = \vec{p}\cdot\vec{F} \qquad E = mc^2$$

$$= m\vec{v}\cdot\vec{F}$$

$$\Rightarrow \boxed{\frac{dE}{dt} = \vec{v}\cdot\vec{F}}$$

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Acceleration is not $\parallel^{l}$ to force always

$$\vec{F} = m\vec{f}$$

$$\Downarrow$$

$$\vec{F} \propto \vec{f}$$

$$\vec{F} = \frac{d\vec{p}}{dt} = m\frac{d\vec{u}}{dt} + \vec{u}\frac{dm}{dt} = m\frac{d\vec{u}}{dt} + \frac{\vec{u}}{c^2}\frac{dE}{dt} \qquad \boxed{E = mc^2}$$

$$= m\frac{d\vec{u}}{dt} + \frac{\vec{u}}{c^2}\frac{d}{dt}(T + m_0c^2)$$

$$= m\frac{d\vec{u}}{dt} + \frac{\vec{u}}{c^2}\frac{d}{dt}(\vec{F}\cdot d\vec{x})$$

$$\vec{F} = m\frac{d\vec{u}}{dt} + \frac{\vec{u}}{c^2}(\vec{F}\cdot\vec{u})$$

$$\Rightarrow \boxed{f = \frac{du}{dt} = \frac{\vec{F}}{m}} - \underbrace{\frac{\vec{u}}{mc^2}(\vec{F}\cdot\vec{u})}_{\Downarrow\ 0}$$

$$\vec{f} \parallel \vec{F}$$

$$\vec{F}\cdot\vec{u}$$

Longitudinal mass / Transverse mass

$$\vec{F} = \frac{d}{dt}(m\vec{v}) = \frac{d}{dt}(m_0\gamma\vec{v}) = m_0\vec{v}\frac{d\gamma}{dt} + m_0\gamma\frac{d\vec{v}}{dt}$$

$$\frac{d\gamma}{dt} = \frac{d}{dt}\left(\frac{1}{\sqrt{1-v^2/c^2}}\right) = \frac{v}{c^2}\gamma^3 f_{\parallel}$$

$$\vec{v}\cdot\vec{v} = v^2 \Rightarrow$$

$$m = \frac{m_0}{\sqrt{1-v^2/c^2}}$$

$$|\vec{p}| = m\vec{v} = \frac{m_0|\vec{v}|}{\sqrt{1-v^2/c^2}}$$

$$2\vec{v}\cdot\vec{f} = \frac{d}{dt}(v^2) = 2vf_{\parallel}$$

comp of $\vec{f} \parallel \vec{v}$

$$\vec{F} = m_0\gamma\frac{d\vec{v}}{dt} + m_0\vec{v}\frac{v}{c^2}\gamma^3 f_{\parallel}$$

$$F_{\parallel} = m_0\gamma f_{\parallel} + m_0\frac{v^2}{c^2}\gamma^3 f_{\parallel}$$

$$F_{\perp} = m_0\gamma f_{\perp} \rightarrow \text{Transverse mass}$$

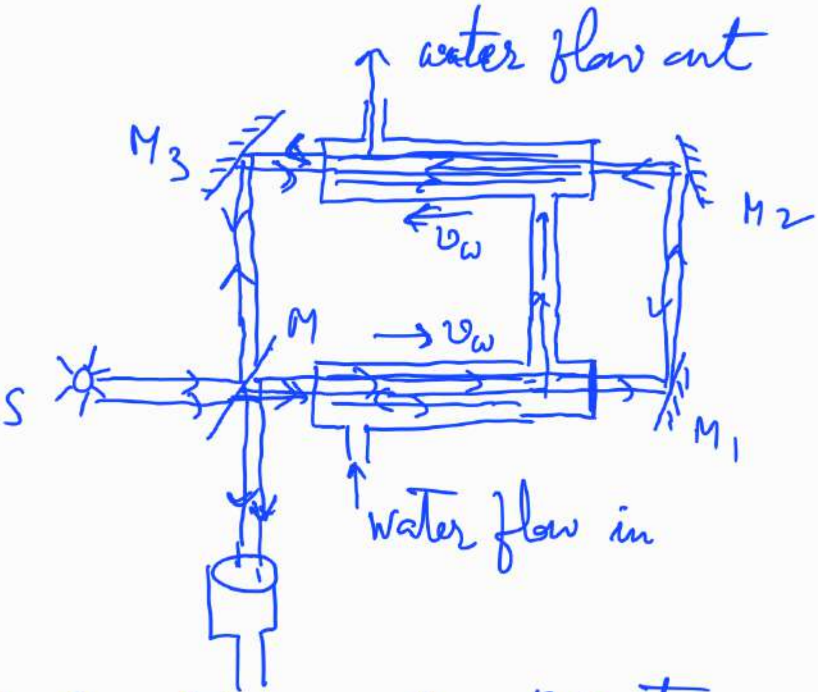
$m_0\gamma$

Longitudinal mass

$$= m_0\gamma + m_0\frac{v^2}{c^2}\gamma^3$$

$$= m_0\gamma^3$$

Fizeau's expt. result. (Ether-drag coeff.)



$u \rightarrow$ vel of water w.r.t. apparatus

$v_x' = \frac{c}{n} \rightarrow$ vel of light w.r.t. water

To an observer at rest w.r.t. water tube the vel of light 'v' is given by

$$v = \frac{v_x' \pm u}{1 \pm \frac{u v_x'}{c^2}}$$

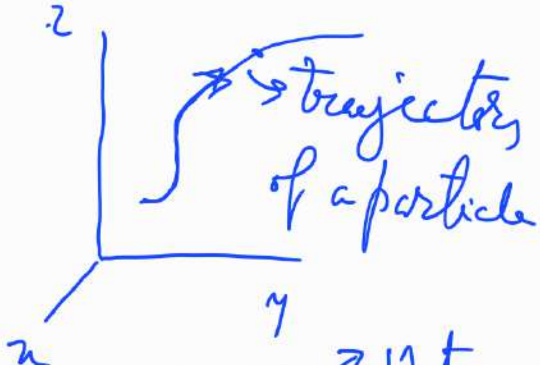
$$= \left(\frac{c}{n} \pm u\right)\left(1 \pm \frac{u}{nc}\right)^{-1}$$

$$\simeq \frac{c}{n} \pm \underbrace{u\left(1 - \frac{1}{n^2}\right)}_{\text{Fresnel's drag coefficient}}$$

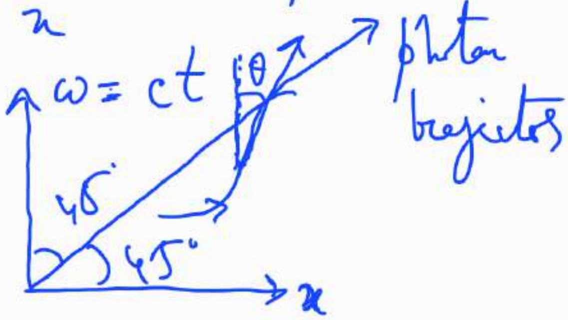
change in vel due to flow of water

$$\Delta v_x = v_x' - v_x$$
$$= u\left(1 - \frac{1}{n^2}\right)$$
$$\simeq \underline{0.444}$$

Geometric representation of space-Time



Minkowski → Teacher of Einstein
$\Downarrow$
4-D al M. space

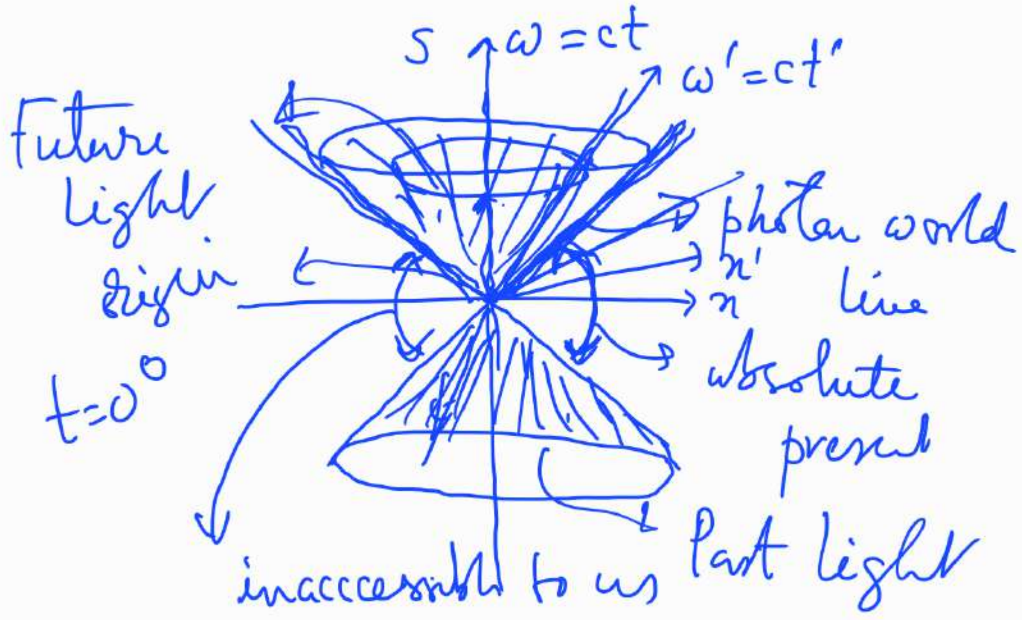


$$x' = \gamma(x - \beta\omega) \quad \omega \to ct$$
$$\omega' = ct' = \gamma(\omega - \beta x)$$
$$\omega = ct$$

$$\frac{dx}{d\omega} = \frac{1}{c}\frac{dx}{dt} = \frac{v}{c} < 1$$

$$\tan\theta = \frac{dy}{dx}$$

For photon $\frac{dx}{d\omega} = \tan\theta$
$$= \tan 45 = 1$$



$v > c$

S: (x, y, z, t) (x_1, y_1, z_1, t_1)

$$ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2) = \text{L.I}$$

$$= c^2 dt'^2 - (dx'^2 + dy'^2 + dz'^2)$$

$$dx' = \gamma(dx + v\,dt)$$
$$dt' = \gamma\left(dt + \frac{v\,dx}{c^2}\right)$$

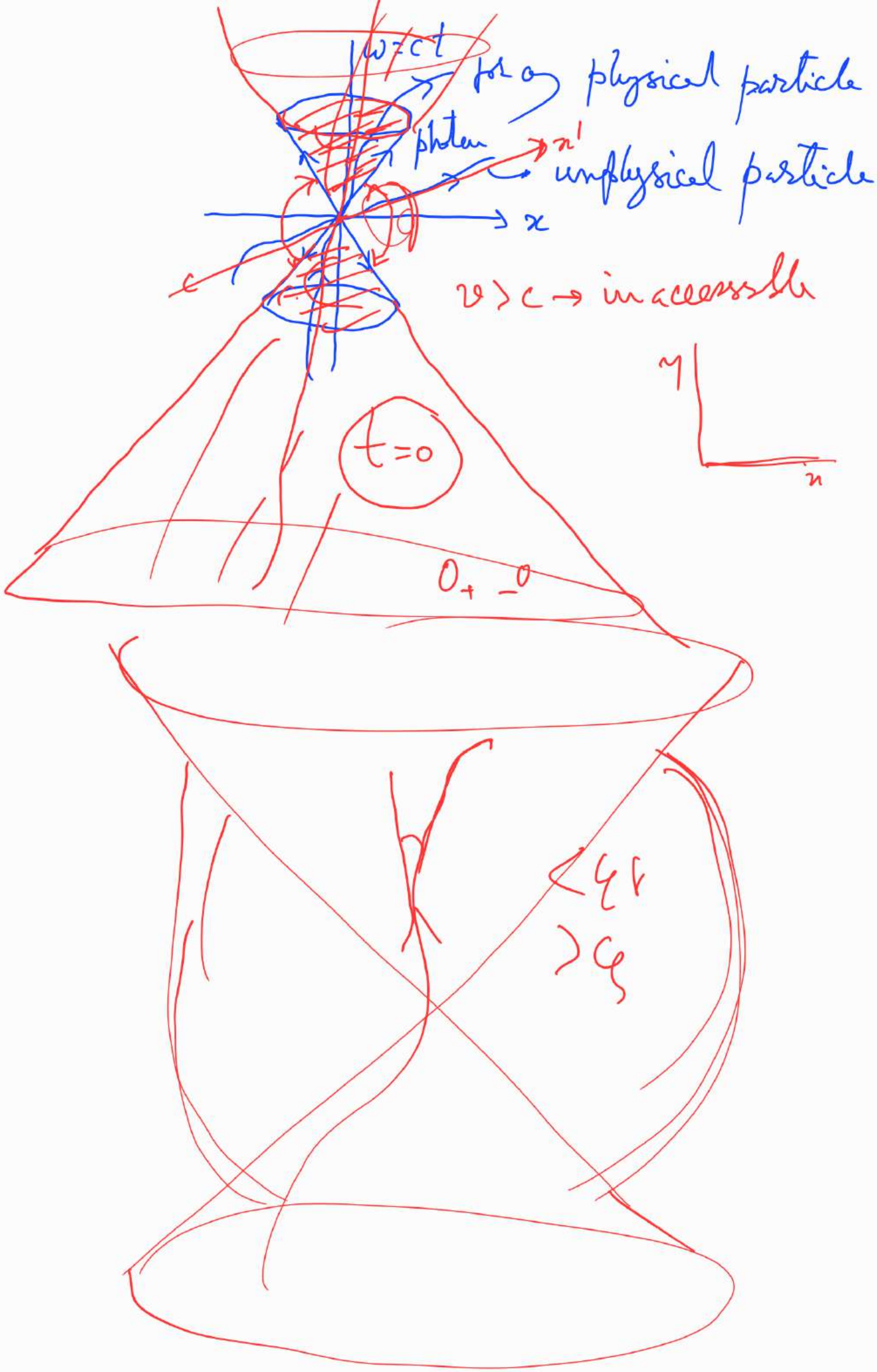
$$d\xi^2 = dx^2 + dy^2 + dz^2 \rightarrow +ve$$

$$ds^2 = c^2 dt^2 - (|d\vec{x}|) \begin{matrix} > 0 \\ < 0 \\ = 0 \end{matrix}$$

$ds^2 > 0 \Rightarrow c > v \rightarrow$ Time like interval

$< 0 \Rightarrow c < v \rightarrow$ Space like " X not allowed in relativity

$= 0 \Rightarrow c = v \rightarrow$



w=ct
for any physical particle
photon
x'
unphysical particle
x
v>c → inaccessible
t=0

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Vector Analysis - Spiegel

$$\vec{p} = m\vec{f} \Rightarrow \frac{d\vec{p}}{dt} = \vec{F} \qquad \frac{dp'}{dt'} = \vec{F}'$$

$S \qquad S'$

Tensor is an object / mathematical structure that transforms properly under coordinate transformation.

Subscript / Superscript:

Dimension of space $= n =$ no of coordinates
rank of Tensor $= r$ total no of indices

Component of tensor $= n^r$

$\vec{A} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}$

① Vector $\to n^1 = n$ compon

$- 4$

Sca

M.I. $\to \mathbb{I} =$ 2nd rank Tensor

$\Downarrow$ $\qquad$ Comp. $= 3^2 = 9$

2nd

$\vec{A}$

Superscript index is used to indicate contravariant character

Subscript " " covariant "

$A^{\mu}{}_{\nu}$ $\mu \rightarrow$ superscript index

$\nu \rightarrow$ subscript "

↳ mixed tensor of rank 2"

$A^{\mu} \rightarrow$ 1st rank tensor contravariant

$A_{\mu} \rightarrow$ covariant

$A^{\mu\nu\alpha\beta}_{\gamma\delta} \rightarrow$ mixed tensor of rank '6'

dimension 3 : $3^6 =$

<u>For notation of coordinates superscript index to be used</u>

x_i ≠ allowed not

$\bar{x}^i \quad i = 1 \cdots N$

↳ $\bar{x}^1, \bar{x}^2, \bar{x}^3 \cdots \bar{x}^N$ (S')

$(x\ y, z, ct)$

$x^j, \ j = 1, \cdots N \rightarrow (S)$

$\bar{A} \leftrightarrow A$

$\bar{A}$

A

A

Contraveriant tensor of rank-1

$A^1, A^2, A^3 \cdots A^N \rightarrow$ no. of components of a quantity in one coordinate system $(x^1, x^2, \cdots x^N)$

$\bar{A}^1, \bar{A}^2, \bar{A}^3 \cdots \bar{A}^N \cdots (\bar{x}^1, \bar{x}^2, \bar{x}^3 \cdots \bar{x}^N)$

$$\bar{A}^\alpha = \sum_{\beta=1}^{N} \frac{\partial \bar{x}^\alpha}{\partial x^\beta} A^\beta = \frac{\partial \bar{x}^\alpha}{\partial x^\beta} A^\beta$$

$$\frac{\partial \bar{x}^\alpha}{\partial x^\beta} \qquad \bar{x} = \bar{x}(x)$$

spherical pole $\qquad$ cartesian

$$\bar{x}^1 = x = r \sin\theta \cos\varphi$$
$$\bar{x}^2 = y = r \sin\theta \sin\varphi$$
$$\bar{x}^3 = z = r \cos\theta$$

$$\bar{A}_\alpha = \sum_{\beta=1}^{N} \frac{\partial x^\beta}{\partial \bar{x}^\alpha} A_\beta \rightarrow \text{covariant}$$

$$\bar{A}_\alpha = \frac{\partial x^\beta}{\partial \bar{x}^\alpha} A_\beta$$

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$\bar{A}\ (\bar{x}^i, i = 1 .. n) \quad A\ (x^i, i = 1, n)$

$$\bar{A}^\alpha = \frac{\partial \bar{x}^\alpha}{\partial x^i} A^i \rightarrow \text{contravariant}$$

$$\bar{A}_\alpha = \frac{\partial x^i}{\partial \bar{x}^\alpha} A_i \rightarrow \text{covariant}$$

2nd rank Tensor

$\bar{x}^\alpha, x^i$

$$\bar{A}^{\alpha\beta} = \frac{\partial \bar{x}^\alpha}{\partial x^i} \frac{\partial \bar{x}^\beta}{\partial x^j} A^{ij} \rightarrow \text{contravari}$$

$$\bar{A}_{\alpha\beta} = \frac{\partial x^i}{\partial \bar{x}^\alpha} \frac{\partial x^j}{\partial \bar{x}^\beta} A_{ij} \rightarrow \text{covariant}$$

$$\bar{A}^\alpha_\beta = \frac{\partial \bar{x}^\alpha}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^\beta} A^i_j$$

Total compr $= n^r$
$= n^2$

Metric Tensor :

$R^3 \rightarrow$ ordinary 3D std cartesian system

$$ds^2 = dx^2 + dy^2 + dz^2 = \Sigma (dx^i)^2 = dx^i dx_i$$
$$= dx^i dx^i = g_{ij} dx^i dx^j$$

$(dx^1)^2 + (dx^2)^2 + (dx^3)^2$ metric tensor

$g_{11} = 1 \quad g_{22} = 1 \quad g_{33} = 1$

$g_{ij} = 0 \quad i \neq j \quad g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$u^i = u^i(x^1, x^2, x^3) \quad , \quad x^i = x^i(u^1, u^2, u^3)$$

$$ds^2 = \frac{\partial x^i}{\partial u^j}\frac{\partial x^i}{\partial u^k} du^j du^k = g_{jk} du^j du^k$$

$$g_{jk} = \frac{\partial x^i}{\partial u^j}\frac{\partial x^i}{\partial u^k} \rightarrow \text{metric tensor in curvilinear space}$$

2nd rank covariant ~~te~~ symmetric

$$g_{jk} = g_{kj}$$

$$d\bar{s}^2 = ds^2$$

$$\bar{g}_{ij} d\bar{x}^i d\bar{x}^j = g_{mn} dx^m dx^n \qquad dx^m = \frac{\partial x^m}{\partial \bar{x}^i} d\bar{x}^i$$

$$= g_{mn} \frac{\partial x^m}{\partial \bar{x}^i}\frac{\partial x^n}{\partial \bar{x}^j} d\bar{x}^i d\bar{x}^j$$

$$\Rightarrow \bar{g}_{ij} = g_{mn} \frac{\partial x^m}{\partial \bar{x}^i}\frac{\partial x^n}{\partial \bar{x}^j}$$

$\hookrightarrow$ covariant transformation rule for g_{ij}

<u>Conjugate metric tensor</u> : $|g_{ij}| = g \neq 0$

$$g^{ij} = \frac{\text{cof}(g_{ij})}{g}$$

<u>Cylindrical system</u> : $g_{ij} = ?$

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

$$g_{11} = 1 \quad g_{22} = r^2, \quad g_{33} = 1$$

$$g_{ij}^{\text{cylindrical}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad g_{ij} = 0 \quad i \neq j$$

Spherical polar : $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$

$$g_{11} = 1 \,,\; g_{22} = r^2 \,,\; g_{33} = r^2 \sin^2\theta$$

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

For arbitrary space

$$ds^2 = 2(dx^1)^2 + 3(dx^2)^2 + (dx^3)^2 - 6\,dx^1 dx^2 + 4\,dx^2 dx^3$$

$$g_{11} = 2 \,,\; g_{22} = 3 \,,\; g_{33} = 1$$

$$g_{12} = g_{21} = -3 \,,\; g_{23} = g_{32} = 2$$

$$g_{13} = g_{31} = 0$$

$$g_{ij} = \begin{pmatrix} 2 & -3 & 0 \\ -3 & 3 & 2 \\ 2 & 0 & 1 \end{pmatrix}$$

Riemannian space

$g_{ij} \neq 0 \quad i \neq j$

Raising & Lowering of index.

$$A_i = g_{ij} A^j \Rightarrow A^i = g^{ij} A_j$$

$$g^{\alpha\beta} g_{\gamma\beta} = \delta^\alpha_\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Metric in Minkowski space

$$ds^2 = c^2dt^2 - (dx^2 + dy^2 + dz^2)$$
$$= -c^2dt^2 + (dx^2 + dy^2 + dz^2)$$

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times$$

Time like / vector / interval is only accessible as ($v < c$ here)

if $ds^2 > 0$ is taken for time like interval

then g_{ij} = 1st kind

if $ds^2 < 0$ for time like interval then

$g_{ij} \Rightarrow$ 2nd kind

$$\bar{A}^{\alpha\beta\gamma}_{\delta\mu} = \left(\frac{\partial \bar{x}^{\alpha}}{\partial x^i}\right)\left(\frac{\partial \bar{x}^{\beta}}{\partial x^j}\right)\left(\frac{\partial \bar{x}^{\gamma}}{\partial x^k}\right)\left(\frac{\partial x^m}{\partial \bar{x}^{\mu}}\right) A^{ijk}_{mn} \left(\frac{\partial x^n}{\partial \bar{x}^{\delta}}\right)$$

↳ 5-rank mixed tensor

3/5/2021 :

Velocity is contravariant vector /
gradient covariant vector
$\vec{\nabla}\varphi$

$$\begin{cases} x^i = x^i(\bar{x}^j) & i = 1, n \,,\; j = 1, n \\ \bar{x}^\alpha = \bar{x}^\alpha(x^i) & i = 1, n \quad \alpha = 1, n \end{cases}$$

$\hookrightarrow \quad d\bar{x}^\alpha = \dfrac{\partial \bar{x}^\alpha}{\partial x^i} dx^i$

Velocity in barred coordinate system

$$\bar{v}^\alpha = \frac{d\bar{x}^\alpha}{dt} = \frac{\partial \bar{x}^\alpha}{\partial x^i}\frac{dx^i}{dt} = \frac{\partial \bar{x}^\alpha}{\partial x^i} v^i$$

$\bar{v}^\alpha = \dfrac{\partial \bar{x}^\alpha}{\partial x^i} v^i \rightarrow$ contravariant transformation rule

$$\underbrace{\varphi(x^i) = \bar{\varphi}(\bar{x}^\alpha)}_{\text{condition of scalar}} = \varphi(\bar{x}^\alpha)$$

$$A_i = \frac{\partial \varphi}{\partial x^i} \quad \text{or} \quad \bar{A}_\alpha = \frac{\partial \varphi}{\partial \bar{x}^\alpha}$$

$$\frac{\partial \varphi}{\partial x^i} = \frac{\partial \varphi}{\partial \bar{x}^\alpha}\frac{\partial \bar{x}^\alpha}{\partial x^i} \Rightarrow A_i = \bar{A}_\alpha \frac{\partial \bar{x}^\alpha}{\partial x^i} \Rightarrow \bar{A}_\alpha = \frac{\partial x^i}{\partial \bar{x}^\alpha} A_i$$

covariant transformation rule

Four-Vector ⇒ Vector in 4 Dimension

$(3+1)$ → time dimension

↳ space dimension

$$|\vec{x}|^2 = x^2+y^2+z^2$$

→ invariant w.r.t. 3D^{al} rotation

$$s^2 = c^2\tau^2 = c^2t^2 - (x^2+y^2+z^2)$$

⇓ ↳ L. I. in 4D^{al} Minkowski

Non-Euclidean

$i = \sqrt{-1}$

$$g_{ij} = \eta_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

↓

Minkowski Metric tensor → fundamental metric in Minkowski space

$$g_{ij} = g^{ij} \qquad g_{ij} = g_{ji}$$

$$s^2 = \eta_{\mu\nu} x^\mu x^\nu$$

Any quantity a^μ having 4-components (a^0, a^1, a^2, a^3) where $a^0 = ct$, Time component, $a^i \equiv \vec{a}$ are spatial components of usual 3-vector $\vec{a}$, if transforms under L.T. from one inertial system to another system like

$$a'^{\mu} = a^{\mu}_{\nu}\, a^{\nu} \equiv \Lambda^{\mu}_{\nu}\, a^{\nu}$$

$$a^{\mu}_{\nu} = \Lambda^{\mu}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a'^0 \\ a'^1 \\ a'^2 \\ a'^3 \end{pmatrix} = \begin{pmatrix} & & & \\ & \Lambda^{\mu}_{\nu} & & \\ & & & \end{pmatrix} \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix}$$

$$a'^0 = \gamma a^0 - \beta\gamma a^1 = \gamma(ct - \beta x)$$

$$a'^1 = -\beta\gamma a^0 + \gamma a^1 = \gamma(x - \beta ct)$$

$$\Rightarrow \quad ct' = \gamma(ct - \beta x)$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \quad ✓$$

$$x' = \gamma(x - vt) \quad ✓$$

$$a_{\mu} = (a^0, -\vec{a}) \qquad a^{\mu} = (a^0, \vec{a})$$

$$a \cdot a = a^{\mu} a_{\mu} = a^{0\,2} - (a_1 a^1 + a_2 a^2 + a_3 a^3)$$

$$a \cdot b = a^{\mu} b_{\mu} = a^0 b^0 - (a_1 b^1 + a_2 b^2 + a_3 b^3)$$

$$= a^0 b^0 = (a^1 b_1 + a^2 b_2 + a^3 b_3)$$

$$\underline{x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu}} \qquad x'_{\mu} = g_{\mu\rho} x'^{\rho} = g_{\mu\rho} \Lambda^{\rho}_{\ \sigma} x^{\sigma}$$

$$= g_{\mu\rho} \Lambda^{\rho}_{\ \sigma} g^{\sigma\nu} x_{\nu}$$

$$x'_{\mu} = \Lambda_{\mu}^{\ \nu} x_{\nu}$$

$$\Lambda^{\rho}_{\ \mu} \, g_{\rho\sigma} \, \Lambda^{\sigma}_{\ \nu} = g_{\mu\nu}$$

4/5/21

$$\boxed{\Lambda^{\sigma}_{\ \rho} \Lambda^{\rho}_{\ \mu} = \delta^{\sigma}_{\mu}}$$

<u>Position 4-vector</u>

$$x^{\mu} = (ct, x, y, z) \equiv (x^0, x^1, x^2, x^3)$$

$$= (x^0, x^i) = (x^0, \vec{x})$$

Greek indices used to mean index run from $0 \to 4$

english index is used to mean index runs from $1 \to 3$

$$x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu}$$

$s^2 \to$ L.I.

$$x^{\mu} x_{\mu} = x'^{\mu} x'_{\mu}$$

$$\Rightarrow \; g_{\mu\nu} x^{\mu} x^{\nu} = g_{\sigma\tau} x'^{\rho} x'^{\tau}$$

$$= g_{\rho\tau} \Lambda^{\rho}_{\ \mu} \Lambda^{\tau}_{\ \nu} x^{\mu} x^{\nu}$$

$$\Rightarrow g_{\mu\nu} = g_{\rho\tau} \Lambda^{\rho}_{\ \mu} \Lambda^{\tau}_{\ \nu} \quad \Rightarrow \boxed{g = \Lambda^{T} g \Lambda}$$

$$g^{\mu\sigma} g_{\sigma\nu} = \delta^{\mu}_{\nu}$$

Velocity 4-vector

$$U^\mu = \frac{dx^\mu}{d\tau} \qquad d\tau \to \text{proper time interval}$$

$$d\tau = dt\left(1-\frac{v^2}{c^2}\right)^{1/2}$$

$$\Rightarrow \boxed{dt = \gamma_v\, d\tau}$$

$$U^0 = \frac{dx^0}{d\tau} = \frac{c\,dt}{d\tau} = c\gamma_v$$

$$U^i = \frac{dx^i}{d\tau} = \frac{dx^i}{dt}\frac{dt}{d\tau} = \gamma_v v^i$$

$$U^\mu = \gamma_v \begin{pmatrix} c \\ \vec{v} \end{pmatrix} = \gamma_v(c, \vec{v}) = \gamma_v(c, v^i)$$

$$U_\mu = \gamma_v(c, -\vec{v}) = \gamma_v(c, -v^i)$$

$$U^\mu U_\mu = \gamma_v^2(c^2 - \vec{v}\cdot\vec{v}) = \gamma_v^2(c^2 - v^2)$$

$$= \frac{c^2}{c^2-v^2}(c^2-v^2) = c^2$$

$$U^\mu U_\mu = c^2 = \text{L.I.}$$

$> 0 \to$ time like vector

$$g_{\mu\nu} = +1 \quad \mu=\nu=0$$
$$= -1 \quad \mu=\nu\neq 0$$
$$= 0 \quad \mu\neq\nu$$

$$x'^\mu = \Lambda^\mu{}_\nu\, x^\nu$$

$$\hookrightarrow \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Downarrow$$

$$U'^\mu = \Lambda^\mu{}_\nu\, U^\nu$$

$$U'^0 = \gamma(U^0 - \beta U^1) \quad U'^1 = \gamma(-\beta U^0 + U^1), \quad U'^2 = U^2, \quad U'^3 = U^3$$

$$\Rightarrow \gamma_{u'} c = \gamma(c\gamma_u - \beta\gamma_u u^1) \Rightarrow \gamma_{u'} = \gamma\gamma_u(1 - vu'/c^2)$$

$$\gamma_{u'} u'^{1} = \gamma(c\gamma_u - \beta\gamma_u u^1) \Rightarrow \gamma_{u'} = \gamma\gamma_u\left(1 - vu^1/c^2\right) \rightarrow (8)$$

$$\gamma_{u'} u'^{1} = \gamma\left(-\beta\gamma_u c + \gamma_u u^1\right) \Rightarrow \gamma_{u'} u'^{1} = \gamma\gamma_u\left(u^1 - v\right) \rightarrow (9)$$

$$\boxed{u'^{1} = \frac{u^1 - v}{1 - \frac{u^1 v}{c^2}}}$$

<u>Four momentum:</u>

$$p^\mu = m_0 U^\mu = m_0 \gamma_u (c, u^i)$$

$$= (mc, mu^i) = \left(\frac{E}{c}, p^i\right)$$

<u>$\vec{p}\cdot\vec{p}$</u>

$$= \left(\frac{E}{c}, \vec{p}\right) \qquad p_\mu = \left(\frac{E}{c}, -\vec{p}\right)$$

$$p^\mu p_\mu = p\cdot p = (E/c)^2 - \vec{p}^{\,2} = \frac{E^2 - p^2c^2}{c^2} = m_0^2 c^2$$

$$= \text{L.I.}$$

<u>Acceleration 4-vector:</u> $a^\mu = \dfrac{dU^\mu}{d\tau}$

$$a^0 = \frac{dU^0}{d\tau} = \frac{d}{d\tau}(\gamma_u c) = \gamma_u \frac{d}{dt}(\gamma_u c) \qquad \left[d\tau = \frac{dt}{\gamma_u}\right.$$

$$= \frac{\gamma_u^4}{c} u^i a^i \qquad \dot{\gamma}_u = \frac{\gamma_u^3}{c^2}(\vec{u}\cdot\vec{a})$$

$$a^i = \frac{dU^i}{d\tau} = \frac{d}{d\tau}(\gamma_u u^i) = \gamma_u \frac{d}{dt}(\gamma_u u^i)$$

$$= \gamma_u\left[\gamma_u a^i + u^i(\vec{u}\cdot\vec{a})\frac{\gamma_u^3}{c^2}\right]$$

$$= \gamma_u\left[\underline{\frac{\gamma_u^3}{c}(\vec{u}\cdot\vec{a})},\ \underline{\gamma_u \vec{a} + \vec{u}(\vec{u}\cdot\vec{a})\frac{\gamma_u^3}{c^2}}\right]$$

Acceleration is space like vector

$$U^\mu U_\mu = U\cdot U = c^2 = \text{L.I. const}$$

$$2U\cdot\frac{dU}{d\tau} = 0 \Rightarrow U\cdot a = 0 \Rightarrow a \rightarrow \text{space-like}$$

$$\Rightarrow a^\mu U_\mu = 0$$

Four force: Minkowski force. K^μ $\vec{F}$

$$K^\mu = \frac{dp^\mu}{d\tau} \Rightarrow K^\mu U_\mu = \frac{dp^\mu}{d\tau}U_\mu = \frac{dp^0}{d\tau}U_0 + \frac{dp^i}{d\tau}u_i\gamma_u$$

$\Downarrow$

Relativistic for 2nd law

$$U^\mu = (\gamma_u c, \gamma_u u^i)$$

$$U_\mu = (\gamma_u c, -\gamma_u u^i)$$

$$\Rightarrow K^0U_0 + K^iU_i = \frac{dp^\mu}{d\tau}U_\mu = m_0\frac{dU^\mu}{d\tau}U_\mu = m_0 a^\mu U_\mu = 0$$

$$\frac{dp^\mu}{d\tau}U_\mu = \frac{d}{d\tau}\left(m_0\gamma_u c,\ m_0\gamma_u u^i\right)U_\mu$$

$$K^0U_0 = \frac{d}{d\tau}(m_0\gamma_u c)U_0 = -K^iU_i = -\frac{dp^i}{d\tau}u_i\gamma_u$$

$$[u_i = -u^i]$$

$$= +\frac{d}{d\tau}(m_0u^i)u^i\gamma_u$$

$$\Rightarrow \frac{d}{d\tau}(m_0\gamma_u c)(\gamma_u c) = \frac{d}{d\tau}(m_0u^i)u^i\gamma_u$$

$$\Rightarrow \frac{d}{d\tau}(m_0\gamma_u c^2) = \frac{dp^i}{d\tau}u^i \Rightarrow \gamma_u\frac{d}{d\tau}\left(\frac{m_0c^2}{\sqrt{1-u^2/c^2}}\right) = \gamma_u\frac{dp^i}{d\tau}u^i$$

$$\Rightarrow \frac{dE}{dt} = \vec{F}\cdot\vec{u} \Rightarrow \vec{u}\cdot\frac{d}{dt}(m\vec{u}) = \vec{u}\cdot\left(\frac{dm}{dt}\vec{u} + m\frac{d\vec{u}}{dt}\right)$$

$E = mc^2$

Transformation of momentum 4-vector

$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} \qquad v'^{\mu} = \Lambda^{\mu}_{\nu} v^{\mu}$

$$p'^{\mu} = \Lambda^{\mu}_{\nu} p^{\nu} \checkmark$$

$p_x' = \gamma_v \left(p_x - \frac{Ev}{c^2} \right)$, $p_x = m u_x \quad m = \gamma_u m_0 = \frac{E}{c^2}$

$p_y' = p_y \qquad p_z' = p_z$

$$p^{0\prime} = m_0 \gamma_{u'} c = m_0 c \frac{1}{\left(1 - \frac{u'^2}{c^2}\right)^{1/2}} = m_0 c \frac{\left(1 - \frac{u_x v}{c^2}\right)}{\left(1 - \frac{u^2}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)^{1/2}}$$

$$= mc \frac{1 - \frac{u_x v}{c^2}}{1 - v^2/c^2} = \gamma_v \left(mc - \frac{m u_x v}{c} \right)$$

$$= \gamma_v \left(\frac{E}{c} - \frac{p_x v}{c} \right) \qquad m = m_0 \gamma_u \quad p_x = m u_x$$

$p^{0\prime} = \gamma_v \left(p^0 - \beta p^1 \right)$, $p^0 = \frac{E}{c}$, $p^1 = p_x$

$p'^1 = \gamma_v \left(p^1 - \beta p^0 \right)$

05/5/21

Problem Soln in Tensorial Part

1. Find the metric of a coordinate system (a, b, c) which is related to the Cortesian system by

$$x = bc, \quad y = ca, \quad z = ab$$

$$(x, y, z) \quad (a, b, c) \qquad ds^2\big|_{a,bc} = ?$$

$$x = bc \Rightarrow dx = b(dc) + c(db)$$

$$y = ca \Rightarrow dy = a(bc) + c(da)$$

$$z = ab \Rightarrow dz = a(db) + b(da)$$

$$ds^2\big|_{a,b,c} = (dx)^2 + (dy)^2 + (dz)^2$$

$$= (b^2 + c^2)(da)^2 + (c^2 + a^2)(db)^2 + (a^2 + b^2)(dc)^2$$

$$+ \; 2ca\,(da)(dc) + 2bc\,(db)(dc) + 2ab\,(da)(db)$$

$$g_{\mu\nu} = \begin{pmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{pmatrix}$$

2. $A^{\mu\nu}, \; B^{\mu\nu}, \quad A^{\mu\nu} B_{\mu\nu} = A_{\mu\nu} B^{\mu\nu}$

$$A^{\mu\nu} = A_{\alpha\beta}\, g^{\alpha\mu} g^{\beta\nu}, \quad B_{\mu\nu} = B^{\rho\gamma} g_{\rho\mu} g_{\gamma\nu}$$

$$A^{\mu\nu} B_{\mu\nu} = A_{\alpha\beta} B^{\rho\gamma} g^{\alpha\mu} g_{\rho\mu} g^{\beta\nu} g_{\gamma\nu}$$

$$= A_{\alpha\beta} B^{\rho\gamma} \delta^{\alpha}_{\rho} \delta^{\beta}_{\gamma} = A_{\alpha\beta} B^{\alpha\gamma} \delta^{\beta}_{\gamma} = A_{\alpha\beta} B^{\alpha\beta}$$

$$\Rightarrow A^{\mu\nu} B_{\mu\nu} = A_{\mu\nu} B^{\mu\nu}$$

3. Examine if A^{μ}_{ν} being a symmetric tensor in one system, remains so in any other system.

$$A^{\mu}_{\nu} = A^{\nu}_{\mu} \qquad x^i, \ \bar{x}^j$$

$$\bar{A}^i_j = \frac{\partial \bar{x}^i}{\partial x^\mu} \frac{\partial x^\nu}{\partial \bar{x}^j} A^\mu_\nu \rightarrow (1)$$

Interchanging (i,j) & (μ,ν)
$$\bar{A}^j_i = \frac{\partial \bar{x}^j}{\partial x^\nu} \frac{\partial x^\mu}{\partial \bar{x}^i} A^\nu_\mu$$
$$= \frac{\partial \bar{x}^j}{\partial x^\nu} \frac{\partial x^\mu}{\partial \bar{x}^i} A^\mu_\nu \rightarrow (2)$$

RHS of (1) & (2) not equal
$$\bar{A}^i_j \neq \bar{A}^j_i$$

(4) A_i, B_j, $C^{ij} A_i B_j \rightarrow$ scalar

$C^{ij} \rightarrow$ 2nd rank contravariant tensor

$$\bar{C}^{ij} \bar{A}_i \bar{A}_j = C^{pq} A_p A_q$$

$$\bar{A}_i = \frac{\partial x^p}{\partial \bar{x}^i} A_p \qquad \left[\bar{C}^{ij} \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} - C^{pq} \right] A_p A_q = 0$$

$$\bar{B}_j = \frac{\partial x^q}{\partial \bar{x}^j} B_q \qquad \Rightarrow C^{pq} = \bar{C}^{ij} \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j}$$

$$\bar{C}^{ij} = C^{pq} \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial \bar{x}^j}{\partial x^q}$$

5. $A^\mu B_\mu \rightarrow$ four vector 4. vector

$$A'_\mu = \Lambda^\nu_\mu A_\nu \qquad B'^\lambda = \Lambda^\lambda_\rho B^\rho$$

$$A'_\mu B'^\mu = \Lambda^\nu_\mu A_\nu \Lambda^\mu_\rho B^\rho = \Lambda^\nu_\mu \Lambda^\mu_\rho A_\nu B^\rho$$

$$= \delta^\nu_\rho A_\nu B^\rho = A_\nu B^\nu$$

Alternative

$$\bar{A}^0 = \gamma(A^0 - \beta A^1) \quad \bar{A}^1 = \gamma(A^1 - \beta A^0) \quad \bar{A}^2 = A^2 \quad \bar{A}^3 = A^3$$

$$\bar{A}^\mu = \underset{\substack{\Downarrow \\ \text{L.T. matrix}}}{\Lambda^\mu_\nu} A_\nu \Rightarrow \begin{pmatrix} \bar{A}^0 \\ \bar{A}^1 \\ \bar{A}^2 \\ \bar{A}^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}$$

$$A^\mu = (A^0, A^1, A^2, A^3)$$

$$A_\mu = (A^0, -A^1, -A^2, -A^3)$$

$$\bar{A}_\mu \bar{B}^\mu = \bar{A}^0 \bar{B}^0 - \bar{A}_1 \bar{B}^1 - \bar{A}_2 \bar{B}^2 - \bar{A}_3 \bar{B}^3$$

$$= \gamma^2 (A^0 - \beta A_1)(B^0 - \beta B^1) - \gamma^2 (A_1 - \beta A^0)(B^1 - \beta B^0)$$
$$- A_2 B^2 - A_3 B^3$$

$$= A^0 B^0 - A_1 B^1 - A_2 B^2 - A_3 B^3$$

$$= A_\mu B^\mu \rightarrow \text{L.I.}$$

6 Component of a covariant vector in Cartesian system

$A_1 = \frac{x}{y}$ $A_2 = \frac{y}{x}$ → Find them in Polar coordinate

$x^1 \equiv x$ $A_1 = \frac{x^1}{x^2}$ $\bar{x}^1 \equiv r$ $x^1 = \bar{x}^1 \cos \bar{x}^2$

$x^2 \equiv y$ $A_2 = \frac{x^2}{x_1}$ $\bar{x}^2 = \theta$ $x^2 = \bar{x}^1 \sin \bar{x}^2$

$\bar{A}_k = \frac{\partial x^j}{\partial \bar{x}^k} A_j$ $\bar{A}_1 = \frac{\partial x^1}{\partial \bar{x}^1} A_1 + \frac{\partial x^2}{\partial \bar{x}^1} A_2 = \sin\theta + \cos\theta$

$\bar{A}_2 = \frac{\partial x^1}{\partial \bar{x}^2} A_1 + \frac{\partial x^2}{\partial \bar{x}^2} A_2 = r \cot\theta \cos\theta - r \tan\theta \sin\theta$

06/5/24

Quantum Mechanics

Fine structure of H-atom

Darwin Correction

Relativistic / Spin-orbit

$$E^{T}_{n,j} = -\frac{13.6}{n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)\right]$$

$$\lambda_c = \frac{\hbar}{mc}$$

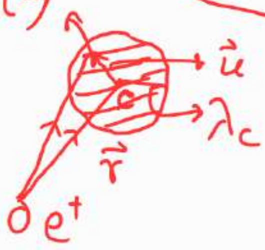
$$V(\vec{r}) = (-e)\varphi(\vec{r}) = -e\,\frac{e}{r},$$

$\varphi(r) \to$ potential created by proton at electron

$$\tilde{V}(\vec{r}) = \int_{\text{electron}} d^3u\, \rho(\vec{u})\,\varphi(\vec{r}+\vec{u}),$$

$\rho(\vec{u})$

$1.61 \times 10^{-19}\varphi$



$$\rho(\vec{u}) = -e\rho_0(u), \quad \int_{\text{electron}} d^3u\, \rho_0(u) = 1$$

$$-e\,\varphi(\vec{r}+\vec{u}) = V(\vec{r}+\vec{u})$$

$$\tilde{V}(\vec{r}) = \int_{\text{electron}} d^3u\, \rho_0(\vec{u})\, V(\vec{r}+\vec{r})$$

$$\rho_0(\vec{u}) = \rho_0(u)$$

$$V(\vec{r}+\vec{u}) = V(\vec{r}) + \sum_i \partial_i V\big|_{\vec{r}} u_i + \frac{1}{2}\sum_{i,j} \partial_i \partial_j V\big|_{\vec{r}} u_i u_j$$

$$\tilde{V}(\vec{r}) = \int d^3u \, \rho_0(u) \left[V(\vec{r}) + \cdots \right]$$

$$= V(\vec{r}) + \sum_i \partial_i V\big|_{\vec{r}} \int d^3u \, \rho_0(u) u_i + \cdots$$

$$\underline{\underline{\delta V}} = \frac{1}{10} u_0^2 \nabla^2 V = \frac{\hbar^2}{10 m^2 c^2} \nabla^2 V \qquad u_0 = \lambda_c = \frac{\hbar}{mc}$$

Correction due to Darwin term

$$H'_{D.T} = -e\delta V = -\frac{e^2\hbar^2}{10 m^2 c^2} \nabla^2 V$$

$$= \frac{\pi}{2} \frac{e^2 \hbar^2}{m^2 c^2} \delta(\vec{r})$$

$\Downarrow$

ns state

$$E'_{D.T.} = \langle \psi_{n00} | H'_{D.T} | \psi_{n00} \rangle = \frac{\pi}{2} \frac{e^2\hbar^2}{m^2c^2} |\psi_{n00}(0)|^2$$

$\Downarrow$

$$\frac{1}{\pi n^3 a_0^3}$$

$$= \alpha^4 mc^2 \frac{1}{2n^3}$$

$\psi_0(0) = 0 \;\; \forall \, l > 0$

$= \text{nonzero} \;\; l = 0$

D.T. affects only s orbital

Atoms in Magnetic & Electric field

Zeeman effect (1896)

⇓

Splitting of spectral line into several components in presence of static magnetic field

Normal Z. effect → $(S=0)$

Anomalous Z. " → $(S \neq 0)$

$$H'_Z = -(\vec{\mu}_L + \vec{\mu}_S) \cdot \vec{B}_{ext}$$

$$\mu_L = -\frac{e}{2m}\vec{L} \qquad \vec{\mu}_S = -\frac{e}{m}\vec{S}$$

$$H'_Z = \frac{e}{2m}(\vec{L} + 2\vec{S}) \cdot \vec{B}_{ext}$$

$$\vec{B}_{int} = \frac{1}{4\pi\epsilon_0}\frac{e}{mc^2 r^3}\vec{L}$$

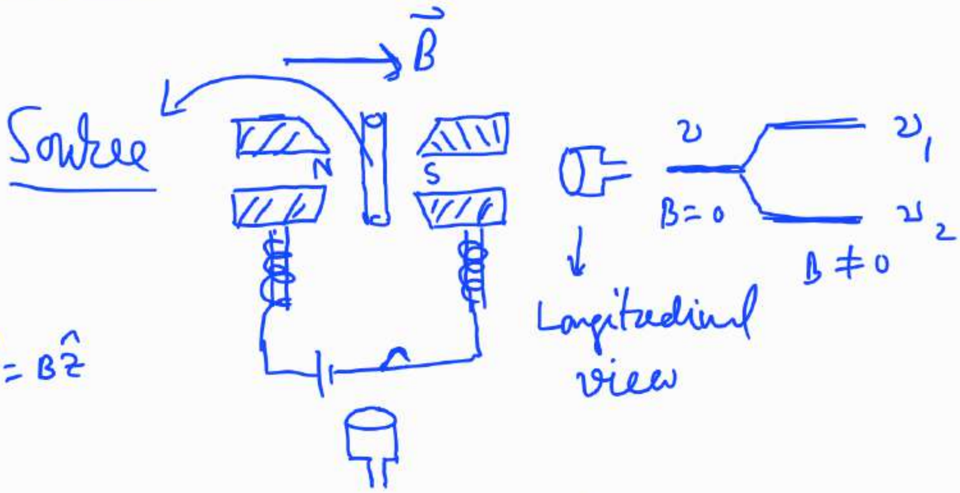
(a) $\vec{B}_{ext} << \vec{B}_{int}$ → fine structure dominates, H'_Z to be treated as small perturbation

(b) $\vec{B}_{ext} >> \vec{B}_{int}$ → Zeeman effect dominates, H'_{SO} to be treated as perturbation → Paschen-Back effect.

(c) $\tilde{B}_{ext} \sim \tilde{B}_{int}$ → Zeeman effect

10/5/21

Normal Zeeman effect $(S=0)$



Theory :

$$\vec{B} = B\hat{z}$$

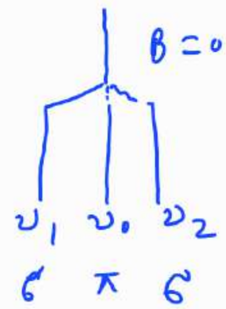
$$L_z = m_l \hbar$$

$$m_l = (2l+1)$$

$$\vec{\mu}_l = -\frac{e}{2m}\vec{l}$$

$$\Delta E = -\vec{\mu}_l \cdot \vec{B}$$

$$= \frac{e\hbar}{2m} m_l B = \frac{ehB}{4\pi m} m_l$$



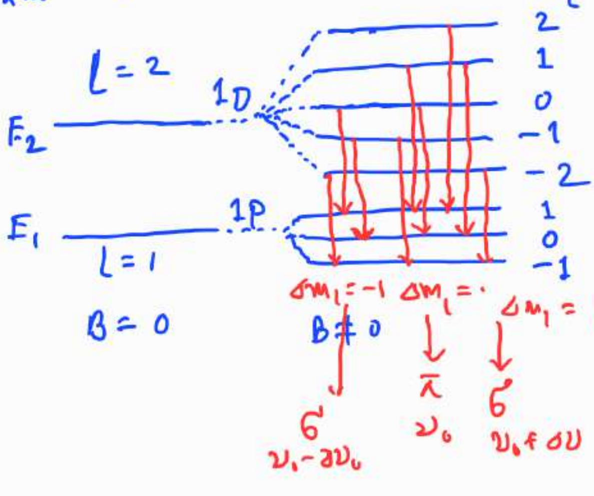
$$\Delta l = \pm 1, \quad \Delta m_l = 0, \pm 1$$

$$h\nu = E_2' - E_1' = E_2 + \mu_B B m_{l_2} - (E_1 + \mu_B B m_{l_1})$$

$$\nu = \frac{E_2 - E_1}{h} + \frac{\mu_B B}{h}(m_{l_2} - m_{l_1}) = \nu_0 + \frac{\mu_B B}{h}\Delta m_l$$

$$\Delta\nu = |\nu - \nu_0| = \frac{eB}{4\pi m}\Delta m_l$$

$S = 0$



$\Delta m_l = 0, \pm 1$

Anomalous Z. effect ($S \neq 0$)

$$E'_Z = \langle n, l, j, m_j | H'_Z | n, l, j, m_j \rangle = \frac{e}{2m} \vec{B}_{ext} \cdot (\vec{L} + 2\vec{S})$$

$$\vec{S}_{av} = \frac{\vec{S}\cdot\vec{J}}{J^2}\vec{J}$$

$$\vec{L} = \vec{J} - \vec{S} \Rightarrow L^2 = J^2 + S^2 - 2\vec{J}\cdot\vec{S}$$

$$\Rightarrow \vec{J}\cdot\vec{S} = \frac{1}{2}(J^2 + S^2 - L^2) = \frac{\hbar^2}{2}[j(j+1) + s(s+1) - l(l+1)]$$

$$\langle(\vec{L} + 2\vec{S})\rangle = \left\langle\left(1 + \frac{\vec{S}\cdot\vec{J}}{J^2}\right)\vec{J}\right\rangle = \left[1 + \frac{j(j+1) - l(l+1) + \frac{3}{4}}{2j(j+1)}\right]\langle\vec{J}\rangle$$

g_J

$$\Rightarrow E'_Z = \mu_B\, g_J\, B_{ext}\, m_j \qquad \mu_B = \frac{e\hbar}{2m} = 5.79 \times 10^{-5}\ eV/T$$

$m_j \to (2j+1)$

$g_J \to$ Lande g-facts

$\Delta l = \pm 1 \quad \Delta m_j = 0, \pm 1$

$\Delta m_j = 0 \to \pi$ line

$\pm 1 \to \sigma$ line

For $Na^{11} \to 1s^2 2s^2 2p^6 3s^1$

Transition between $l = 0$ state (s-state) & $l = 1$ (p-state)

$n = 3,\ l = 0,\ 3s$ state $s = \frac{1}{2} \Rightarrow 2s+1 = 2,\ j = \frac{1}{2}\ (l = 0)$

$\Downarrow$ singlet

Term symbol $\Rightarrow n\,^{2s+1}(l)_j \Rightarrow 3\,^2S_{1/2} \Rightarrow g_J = 2,\ g_J m_j = 2m_j$

$m_j = \pm\frac{1}{2}\ (\because j = \frac{1}{2}) \Rightarrow g_J m_j = \pm 1$

$\Delta E = \pm \mu_B B$

For $n = 3,\ l = 1$ (3p state) $s = \frac{1}{2}\quad 2s+1 = 2,\ j = l \pm \frac{1}{2}$ for $l \neq 0$

$j = \frac{1}{2}, \frac{3}{2}$

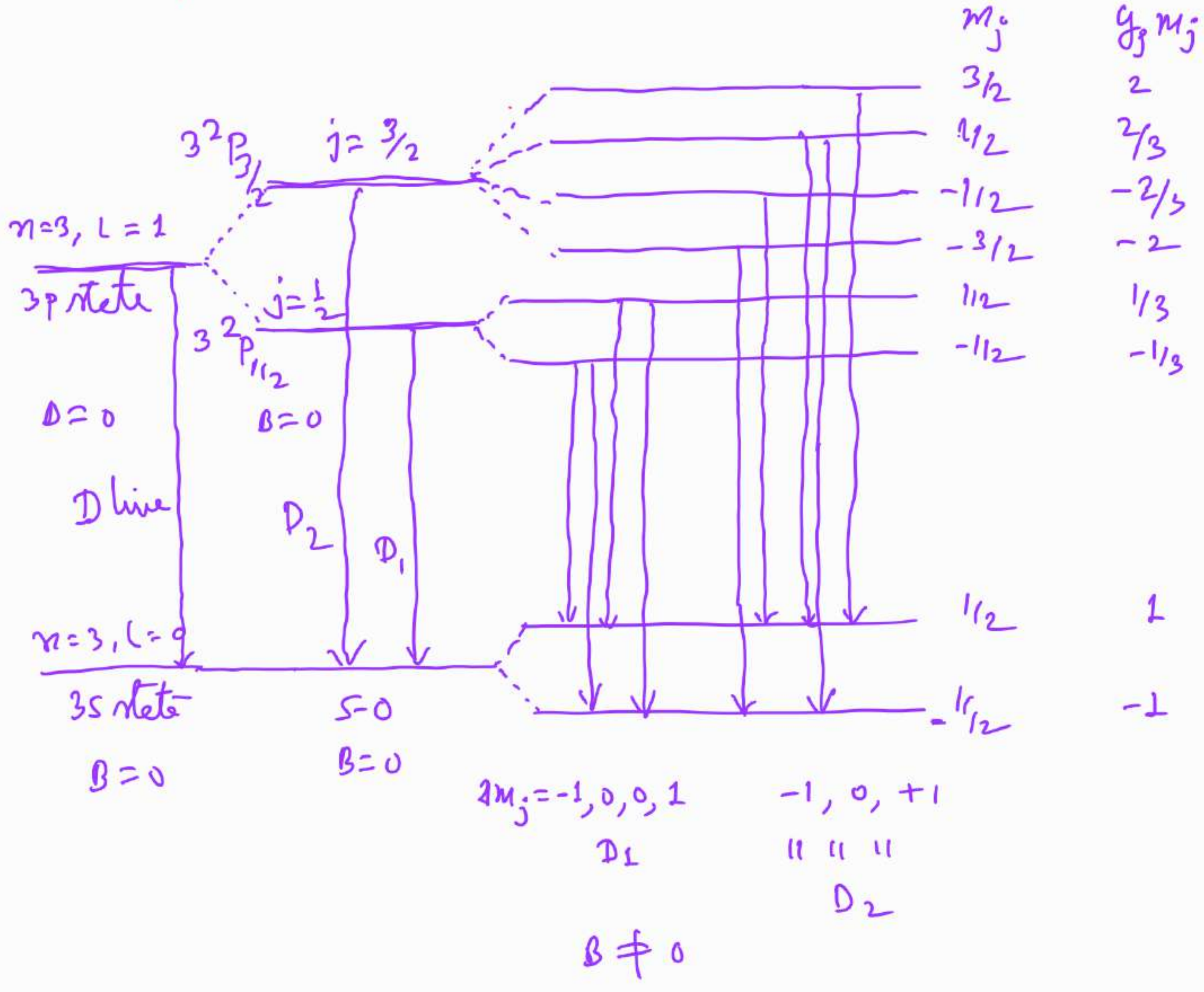
Term symbol $3\,^2P_{1/2}$, $3\,^2P_{3/2}$

$$g_J = \frac{2}{3} \; (j = \tfrac{1}{2}) \quad = \quad \frac{4}{3} \; (j = 3/2)$$

$$\Delta E = \pm \frac{1}{3} \mu_B B \begin{pmatrix} j = \frac{1}{2} \\ l = 1 \end{pmatrix} \qquad \frac{4}{3} m_j \mu_0 B$$

	m_j
$2\mu_B B$	$3/2$
$\frac{2}{3}\mu_B B$	$1/2$
$-\frac{2}{3}\mu_B B$	$-1/2$
$-2\mu_B B$	$-3/2$

$n=3,\ l=0$ (3s)



(1921) Paschen-Back effect $\vec{B}_{ext} \gg \vec{B}_{int}$

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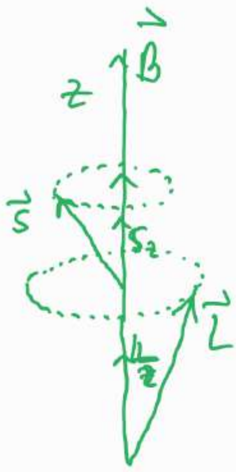
$$\vec{J} = \vec{L} + \vec{S}$$



$$\Delta E_L = -\vec{\mu}_L \cdot \vec{B}$$
$$= -\frac{-e}{2m}\vec{L}\cdot\vec{B}$$
$$= \frac{e}{2m} L_z B$$
$$= \frac{e m_l B \hbar}{2m}$$

$$\Delta E_S = -\vec{\mu}_S \cdot \vec{B}$$
$$= 2\left(-\frac{e}{2m}\left(-\vec{S}\cdot\vec{B}\right)\right)$$
$$= \frac{e}{m} S_z B$$
$$= \frac{e m_s B \hbar}{m}$$

$$\Delta E = (m_l + 2m_s)\frac{e\hbar}{2m} B$$

$$(2l+1)(2s+1)$$

$$\Delta(m_l + 2m_s) = 0, \pm 1$$

$$\Delta m_l = 0, \pm 1$$

$$\Delta l = \pm 1 \qquad \Delta m_s = 0$$

Stark-effect: (1913)

$$E = 10^5 \, V/cm$$

Many electron atoms

identical particles: Symmetric/antisymmetric wavefunction

$$\hat{H}(1,2) = -\frac{\hbar^2}{2m}\left(\nabla_1^2+\nabla_2^2\right) + V(1)+V(2)$$

$$= \hat{H}_1 + \hat{H}_2 \rightarrow (1)$$

$$\psi(1,2) = \psi(1)\psi(2) \rightarrow (2)$$

$$\psi_{\alpha,\beta}(1,2) = \psi_\alpha(1)\psi_\beta(2) \rightarrow (3)$$

$$\psi_{\beta\alpha} = \psi_\beta(1)\psi_\alpha(2) \rightarrow (4)$$

$$\psi_{\alpha\beta}^*\psi_{\alpha\beta} = \psi_\alpha^*(1)\psi_\beta^*(2)\psi_\alpha(1)\psi_\beta(2)$$

$$\psi_{\beta\alpha}^*\psi_{\beta\alpha} = \psi_\beta^*(1)\psi_\alpha^*(2)\psi_\beta(1)\psi_\alpha(2)$$

$$\psi_{\alpha\beta}^*\psi_{\alpha\beta}\Big|_{\substack{1\to 2\\ 2\to 1}} \longrightarrow \psi_\alpha^*(2)\psi_\beta^*(1)\psi_\alpha(2)\psi_\beta(1) = \psi_{\beta\alpha}^*\psi_{\beta\alpha}$$

$$\psi(\vec{r}_1, \vec{r}_2, \cdots \vec{r}_N, s_{z1}, s_{z2}, \cdots s_{zN})$$

$$\hat{H} = -\frac{\hbar^2}{2m}\left(\nabla_1^2+\nabla_2^2+\cdots+\nabla_N^2\right) + V(1,2,3,\cdots N)$$

$$1\to 2 \quad \downarrow$$

$$\hat{H}' = -\frac{\hbar^2}{2m}\left(\nabla_2^2+\nabla_1^2+\cdots+\nabla_N^2\right) + V(2,1,\cdots N)$$

$$\hat{H} = \hat{H}' \qquad [V(2,1 \ldots) = V(1,2 \ldots)]$$

$$\hat{H}\psi(1,2) = E\psi(1,2)$$

$$\hat{H}'\psi(2,1) = E\psi(2,1)$$

$$\psi'(1,2) = A\psi(1,2) + B\psi(2,1)$$

$$|\psi'(1,2)|^2 = |\psi'(2,1)|^2 \Rightarrow A = \pm B$$

$$\psi'(1,2) = A[\psi(1,2) \pm \psi(2,1)]$$

$$\psi_{\alpha\beta}(1,2) = A[\psi_\alpha(1)\psi_\beta(2) \pm \psi_\alpha(2)\psi_\beta(1)]$$

$\rightarrow$ Normalization $\Rightarrow A = \frac{1}{\sqrt{2}}$

$$\psi_{\alpha\beta}(1,2) = \frac{1}{\sqrt{2}}[\psi_\alpha(1)\psi_\beta(2) \pm \psi_\alpha(2)\psi_\beta(1)]$$

$$\psi_{\alpha\beta}(1,2) = \frac{1}{\sqrt{2}}[\psi_\alpha(1)\psi_\beta(2) - \psi_\alpha(2)\psi_\beta(1)]$$

$\Downarrow$ Fermion $\qquad$ (+) for $\rightarrow$ Boson

$\Downarrow$ anti-symmetric $\qquad$ $\Downarrow$ symmetric

For N-particle $\Rightarrow$ $$\psi(1,2 \ldots N) = \frac{1}{\sqrt{N}} \begin{vmatrix} \psi_\alpha(1) & \psi_\alpha(2) & \cdots & \psi_\alpha(N) \\ \psi_\beta(1) & \psi_\beta(2) & \cdots & \psi_\beta(N) \\ \vdots & & & \\ \psi_n(1) & \psi_n(2) & \cdots & \psi_n(N) \end{vmatrix}$$

Slater determinant

Exchange operator $\hat{P}_{12}$

$$\hat{P}_{12}\psi(1,2,\cdots N) = \psi(2,1,3\cdots N)$$

$$\hat{P}_{12}^2\psi = p^2\psi = \psi$$

$$p^2 = 1 \Rightarrow p = \pm 1, \quad \hat{P}_{12}\psi = \pm\psi$$

$$[\hat{H}, \hat{P}_{12}] = 0$$

$$\psi(1,2,3,\cdots N) = \pm\psi(2,1,3\cdots N)$$

$$\psi(1,2\cdots N) = -\psi(2,1,\cdots N)$$

|| spin → repulsion between electrons
→ not Coulomb repulsion

anti ||^l spin → attraction

$||^l \Rightarrow S = 1 \quad 2S+1 = 3, \; \psi_1^+\psi_2^+, \; \psi_1^-\psi_2^-, \; \frac{1}{\sqrt{2}}(\psi_1^+\psi_2^- + \psi_1^-\psi_2^+)$

$$\psi_{space} = \frac{1}{\sqrt{2}}[\psi_a(1)\psi_b(2) - \psi_b(1)\psi_a(2)]$$

<u>Hund's rule / Pauli exclusion principle</u>

$p_x \quad p_y \quad p_z$

$\uparrow\downarrow \quad \uparrow\downarrow \quad \uparrow\downarrow$

$p_x \quad p_y \quad p_z$

$\uparrow\downarrow \quad \downarrow\uparrow \quad \uparrow\downarrow$

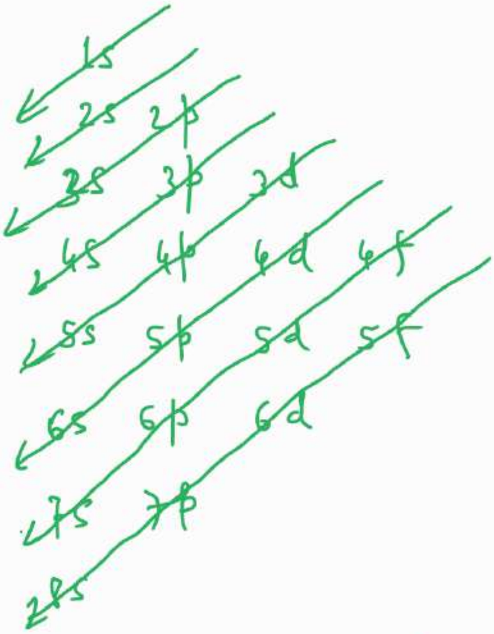
$d_{xy} \quad d_{yz} \quad d_{zx} \quad d_{x^2-y^2} \quad d_{z^2} \to J$

$m_l = 2, 1, 0, -1, -2, \; L = 2$

12/5/24

Aufbau principle

1s < 2s < 2p < 3s < 3p < 4s < 3d < 4p < 5s <



(n, l, s, j, m_j)

$(n, l, s\ m_L, m_S)$

L-S / j-j coupling

L-S - coupling Russell-Sander's

$\Downarrow$

Lighter atom

$$L = l_1 + l_2 + l_3 \cdots$$

$$S = s_1 + s_2 + s_3 \cdots$$

$$\vec{J} = \vec{L} + \vec{S}$$

$(2S+1)$ $L > S$

$(2L+1)$ $S > L$

j-j coupling: Heavier atom

$$j_1 = l_1 + s_1$$

$$j_2 = l_2 + s_2$$

$$\vdots$$

$$j_n = l_n + s_n$$

$$\vec{J} = \vec{j_1} + \vec{j_2} + \cdots$$

Spectral notation of atomic state / Term symbol

$^{2S+1}L_J$

$2S+1 \rightarrow$ spin multiplicity $S \rightarrow$ total spin ang. momentum

$L \rightarrow$ total orbital ang. mom

$J \rightarrow$ total ang. mom $= L+S$

$L \rightarrow$ 0 1 2 3 4 5 6 p s

S P D F G H I

$S \rightarrow 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3$

$(2S+1)$ 1 2 3 4 5 6 7

$2S+1 = 2$ $^2P_{1/2}$ $J = \frac{1}{2}$

$S = \frac{1}{2}$ $L = 1$

$^3P_{3/2}$ $2S+1 = 3$

$L = 1$

$J = 3/2$

Salient features of Alkali spectra

Li $[He] + 2s^1$ K $[Ar] + 4s^1$ Cs $[Xe] + 6s^1$

Na $[Ne] + 3s^1$ Rb $[Kr] + 5s^1$

$S \neq 0$ $\frac{1}{2}$ $\uparrow\downarrow$

$L = 0$

$S = 0$ $J = 0$ $\mu_J = 0$

16|6|2021

Prob - Soln

2

$$S \begin{cases} a, \quad b & \text{Place of occurrence } x_a, x_b \\ \downarrow \quad \downarrow & \\ t_a \quad t_b & t_b > t_a \end{cases}$$

$$S' \Big\{ \; (t_b' - t_a') = \gamma \left[(t_b - t_a) - \frac{v}{c^2} (\underbrace{x_b - x_a}_{0}) \right]$$

$$= \gamma (t_b - t_a)$$

$$(t_b - t_a) < (t_b' - t_a') \qquad \gamma > 1$$

$\Downarrow$

3

distance covered by pion $d = v t$

$d \to$ Lab frame, $v \to 0.99c$, $t \to$ pion's frame

$\Downarrow$

What would be w.r.t. Lab frame

5

$$\frac{m_i}{m_f} = \left(\frac{1+\beta}{1-\beta} \right)^{1/2} \qquad \beta = \frac{v}{c}$$

~~One~~ **1st method**: From conservation of momentum

Rocket's momentum = Radiation momentum

$$\frac{m_f v}{\sqrt{1 - v^2/c^2}} = \left(m_i - \frac{m_f}{\sqrt{1 - v^2/c^2}} \right) c \qquad \left[\begin{aligned} E &= \gamma m c^2 \\ p &= \frac{E}{c} \end{aligned} \right.$$

$\Downarrow$

find $m_f / m_i =$

2nd way : Energy conservation

$$m_i c^2 = \frac{m_f}{\sqrt{1-v^2/c^2}} c^2 + E \text{ (radiation energy)}$$

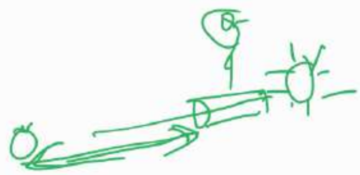
$$0 = \frac{m_f v}{\sqrt{1-v^2/c^2}} - \frac{E}{c}$$

Solving E $\qquad \frac{m_f}{m_i} = ?$

6

7

$$t = \left(\frac{1+\beta}{1-\beta}\right)^{1/2} t'$$

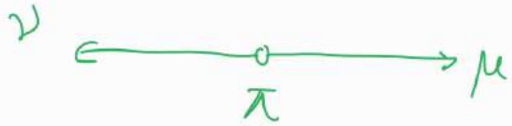


$$S \rightarrow x_1 = (\gamma t')v \quad , \quad t_1 = \gamma t'$$

$$T = \frac{x_1}{c} = \gamma \beta t'$$

$$t = t_1 + T = \gamma t' + \underbrace{\gamma \beta t'}_{\text{time needed for light pulse}} = \sqrt{\frac{1+\beta}{1-\beta}}\, t'$$

8



13

$$m_\pi c^2 = E_\mu + E_\nu$$

$$p_\mu = p_\nu = p \text{ say}$$

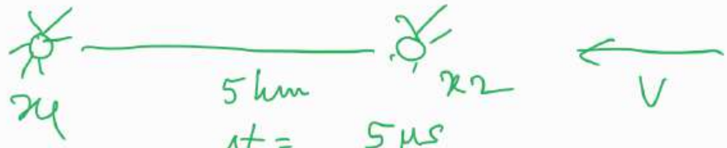
$$E_\mu^2 = p^2c^2 + m_\mu^2 c^4 \qquad E_\nu^2 = p^2c^2 \quad (m_\nu = 0)$$

$$E_\mu^2 - E_\nu^2 = m_\mu^2 c^4 \Rightarrow (E_\mu - E_\nu)(E_\mu + E_\nu) = (E_\mu - E_\nu)\, m_\pi c^2 = m_\mu^2 c^4$$

$$\Rightarrow \quad E_\mu - E_\nu = \frac{m_\mu^2}{m_\pi} c^2 \quad \Rightarrow \quad E_\mu = \frac{c^2}{2m_\pi}\left(m_\pi^2 + m_\mu^2\right)$$

$$E_\mu + E_\nu = m_\pi c^2$$

14



$\Delta t = 5\mu s$

$\Delta x = 5 \text{ km}$

16

photon $\longleftarrow$ π $\longrightarrow$ photon

Pion's momentum $\quad p = \frac{3}{4} m_0 c$

$$E_\pi = \sqrt{p^2c^2 + m_0^2 c^4} = \frac{5}{4} m_0 c^2$$

$\frac{5}{4} m_0 c^2 = E_1 + E_2 \qquad E_1\ E_2$ energy of photon

$$\frac{3}{4} m_0 c = |\vec{p}_1| - |\vec{p}_2| = \frac{E_1}{c} - \frac{E_2}{c}$$

$$\Rightarrow E_1 - E_2 = \frac{3}{4} m_0 c^2$$

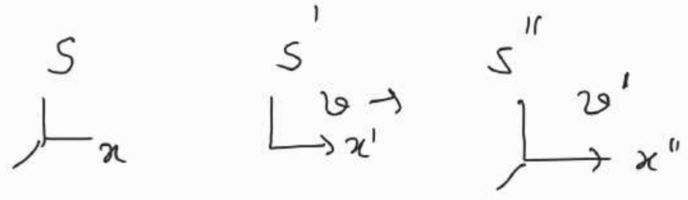
$$E_1 = m_0 c^2 \qquad E_2 = \frac{1}{4} m_0 c^2$$

17

$$\underline{p'^\mu = \Lambda^\mu{}_\nu\, p^\nu} \qquad x'^\mu = \Lambda^\mu{}_\nu\, x^\nu$$

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \qquad \begin{aligned} x' &= \gamma(x - vt) \\ t' &= \gamma\left(t - \frac{vx}{c^2}\right) \end{aligned}$$

17/6/21



$(v+v')/(1+\frac{vv'}{c^2})$

$x' = \gamma(x - vt)$

$x'' = \gamma(x' - vt')$

$x'' = \gamma'(x - vt)$

23

$u^\mu \to$ time like vector

$v^\mu \to$ space like vector $\quad u_\mu v^\mu = 0$

$$u\cdot u = u^\mu u_\mu = (u^0)^2 - \vec{u}\cdot\vec{u} > 0 \Leftarrow [\ g^{\mu\nu} = (+,-,-,-1)$$

$$\cancel{\chi = \vec{u}\cdot\vec{u} - (u^0)^2 > 0}$$

$$x^\mu = (ct, -x, -y, -z)$$

$$(\vec{x}, -ct)$$

$$(u^0)^2 > |\vec{u}|^2$$

$$\Rightarrow |(u^0)| > (|\vec{u}|^2)^{1/2}$$

$$u_\mu v^\mu = 0 \Rightarrow u^0 v^0 = \vec{u}\cdot\vec{v} \Rightarrow \frac{|u^0 v^0|}{|u^0|} = \frac{|\vec{u}\cdot\vec{v}|}{|u^0|} < \frac{|\vec{u}\cdot\vec{v}|}{(|\vec{u}|^2)^{1/2}} \le \frac{|\vec{u}|\cdot|\vec{v}|}{|\vec{u}|}$$

Schwartz inequality

As $|u^0 v^0| = |u^0||v^0|$

$$\Rightarrow |v^0| < |\vec{v}| \Rightarrow (v^0)^2 < \vec{v}\cdot\vec{v}$$

$\Rightarrow v^\mu$ is space like

24

$$\Delta s_{12}^2 = c^2\Delta t_{12}^2 - \Delta x^2 \Rightarrow \text{L.I}$$

$$\Delta x^2 - c^2\Delta t_{12}^2 \longrightarrow \frac{\Delta x}{\Delta t_{12}} > c$$ ↓ spacelike

$$c > \frac{\Delta x}{\Delta t_{12}} \rightarrow$$ time like

$$c \ll \frac{\Delta x}{\Delta t_{12}} \rightarrow$$ space

$$\frac{\Delta x}{\Delta t_{12}} < c$$ ↓ time

$$c^2\Delta t_{12}^2 - \Delta x^2 = c^2\Delta t_{12}'^2 - \Delta x'^2$$

$$S \rightarrow \Delta s_{12}^2 < 0 \longrightarrow$$ space-like

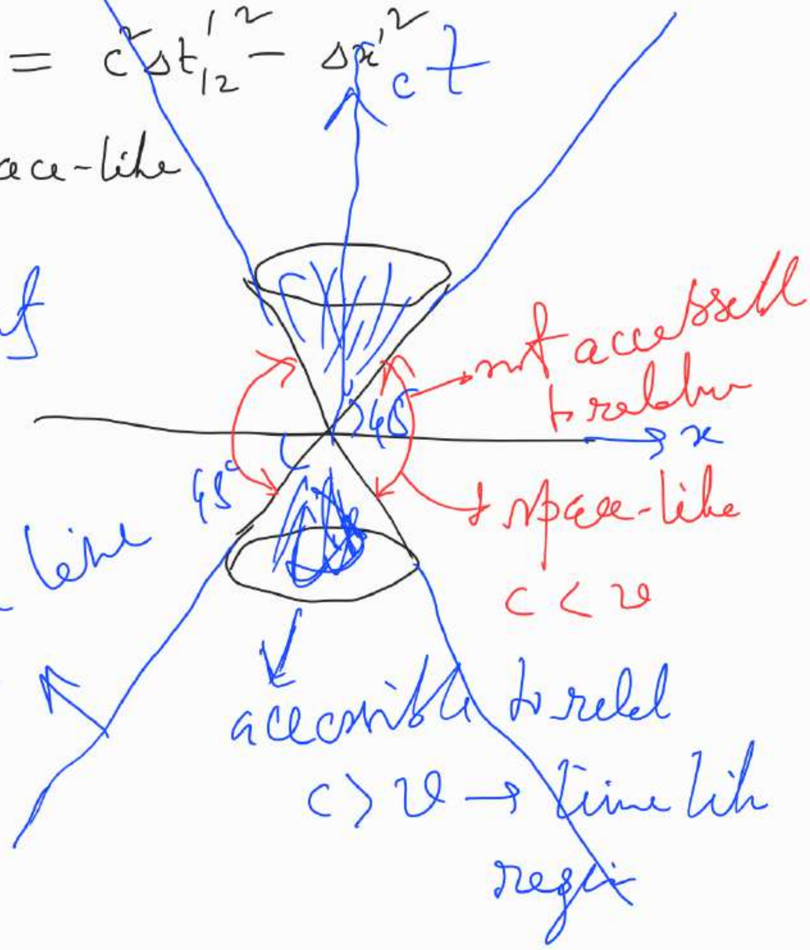
$S' \rightarrow \Delta t'_{12} = 0$ for simultaneity

$$\Delta s'^2_{12} = -\Delta x'^2$$

$$= c^2\Delta t'^2_{12} - \Delta x'^2 < 0$$

$\Delta t'^2_{12} = 0$ ⇓

consistent with space like interval



25

$$K^+ \longrightarrow e^+ + \pi^0 + \nu_e$$

$$E_{e^+}^{max} = ?$$

$m_{K^+} = 494$ MeV $m_{e^+} = 0.5$ MeV

$m_{\pi^0} = 135$ MeV $m_{\nu_e} = 0$

$$p^\mu = (p^0, \vec{p}), \quad p^0 = \frac{E}{c}, \quad p^2 = p^\mu p_\mu = m^2c^2$$

$M, \; m_1 \; m_2, \; m_3$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$K^+ \quad e^+ \quad \pi^0 \quad \nu_e$

$P, \; p_1, \; p_2, \; p_3 \qquad \vec{p}$

$$P = p_1 + p_2 + p_3$$

$$\Downarrow$$

Define $s = P^2 = M^2 c^2$

$$s_1 = (P - p_1)^2 = (p_2 + p_3)^2$$

$\frac{\sqrt{s}}{c} \rightarrow$ invariant mass of Kaon

$\frac{\sqrt{s_1}}{c} \rightarrow$ ------ of system of particles (π^0, ν_e)

$$P^\mu = (Mc, 0, 0, 0)$$

$$s_1 = P^2 + p_1^2 - 2 P^\mu p_{1\mu} = M^2c^2 + m_1^2c^2 - 2ME_1$$

$E_1 = \sqrt{m_1^2 c^4 + \vec{p}_1^{\,2} c^2} \qquad \vec{p}_1 \rightarrow$ momentum of e^+

$$2ME_1 = (M^2 + m_1^2)c^2 - s_1$$

$\Rightarrow E_1$ is maxm when s_1 is minimum

$$s_1 = (p_2 + p_3)^2 = (E_2 + E_3)^2 \geqslant (m_2 + m_3)^2 c^2$$

$$\mathrm{Min}(s_1) = (m_2 + m_3)c^2$$

$$\mathrm{Maxm}(E_1) = \frac{M^2 + m_1^2 - (m_2 + m_3)^2}{2M}$$

= put ~~the~~ values of given mass

27

$$a^\mu a_\mu = (a_x^2 + a_y^2 + a_z^2 - c^2 a_t^2)$$

$$S' \to |a_x^2 + a_y^2 + a_z^2 - c^2 a_t^2| \to |a_x^2 - c^2 a_t^2)$$

$$a_x' = \gamma(a_x - v a_t) = \gamma(a_x - \beta a_0)$$

$$a_t' = \gamma\left(a_t - \frac{v}{c^2} a_x\right) \Rightarrow a_0' = \gamma(a_0 - \beta a_x) \text{ for } a_0 = c a_t$$

If $\beta = \dfrac{a_x}{a_0}$ then $a_x' = 0$ and as $a_0' \neq 0 \to$ time like

$= \dfrac{a_0}{a_x}$ .. $a_0' = 0$ $\quad a_x' \neq 0 \to$ space-like

$a^\mu a_\mu = 0$ for Light-like vect

L.T. means

28

$E \to$ photon $\quad p_e \to$ after photon absorp

$$\frac{E}{c} + 0 = p_e \Rightarrow E = p_e c$$

$$E + m_0 c^2 = (p_e^2 c^2 + m_0^2 c^4)^{1/2} = p_e c + m_0 c^2$$

$$\Rightarrow \underline{p_e^2 c^2 + m_0^2 c^4} + 2 p_e c\, m_0 c^2 = \overline{p_e^2 c^2 + m_0^2 c^4}$$

$\to$ absurd.

impossible

b

(b) $\frac{E}{c} = p_+ + p_- \Rightarrow E = cp_+ + cp_- \quad -①$

$$E = E_+ + E_- = (c^2 p_+^2 + m_0^2 c^4)^{1/2} + (c^2 p_-^2 + m_0^2 c^4)^{1/2} \rightarrow ②$$

→ absurd → impossible

c)

$$e^+_0 + e^- \rightarrow \gamma$$

$p_+ \quad p_- = 0$

$E \; , \; \frac{E}{c}$

$$E = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} - m_0 c^2$$

$$\frac{E}{c} = \frac{m_0 v}{\sqrt{1 - v^2/c^2}}$$

$\frac{v}{c} = 1 \quad v = c$ → absurd impossible

22|6|21:

$$p + p = p + p + p + \bar{p}$$

E, p

$$\begin{cases} E_1 = E + m_0 c^2 & \text{↳ rest energy of stationary proton} \\ p + 0 = p & \rightarrow \text{lab frame} \end{cases}$$

$$E' = 4 m_0 c^2 = (E_1 + E_2)$$

$$E'^2 = E_1^2 - p^2 c^2$$

$$\Rightarrow (4 m_0 c^2)^2 = (E + m_0 c^2)^2 - p^2 c^2 = E^2 - p^2 c^2 + m_0^2 c^4 + 2 E m_0 c^2$$

$$= m_0^2c^4 + m_0^2c^4 + 2E\,m_0c^2$$

$$\Rightarrow E = 7m_0c^2$$

↳ (moving proton)

$m_0c^2 \rightarrow$ rest energy

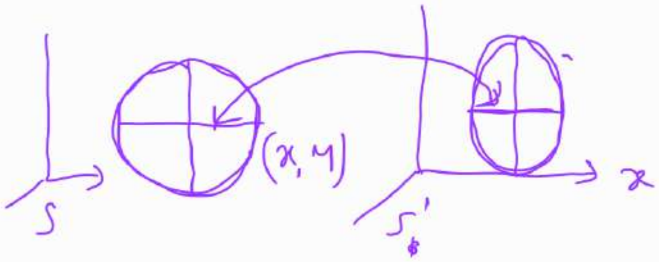
$6m_0c^2$

$\Downarrow$

Incident K.E. of moving proton

$$= 6 \times 938\ \text{MeV} = \underline{5.6\ \text{GeV}}$$

3.



$$(x, y) = (a\cos\theta, a\sin\theta)$$

$$S' \rightarrow (x, y) = (a\gamma^{-1}\cos\theta, a\sin\theta)$$

$$L = L_0\sqrt{1-\beta^2}$$

$$b = a\gamma^{-1}$$

$$L_0 = \frac{L}{\sqrt{1-\beta^2}} = \gamma L$$

$$\epsilon = \left(1 - \frac{b^2}{a^2}\right)^{\frac{1}{2}} = \left(1 - \gamma^{-2}\right)^{1/2}$$

$$= \frac{v}{c}$$

Prob Set - II

(1) $(a, b_c) =$ $x = bc \Rightarrow dx = b(dc) + c(db)$

$y = ca \qquad dy = c(da) + a(dc)$

$z = ab \qquad dz = a(db) + b(da)$

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$= (\quad)(da)^2 + (\quad)(db)^2 + (\quad)(dc)^2$$

$$g_{ij} = \begin{pmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ca & bc & a^2+b^2 \end{pmatrix}$$

2. $A^{\mu\nu}, B^{\mu\nu}$ $\qquad A^{\mu\nu} B_{\mu\nu} = A_{\mu\nu} B^{\mu\nu}$

$$A^{\mu\nu} = A_{\alpha\beta}\, g^{\alpha\mu} g^{\beta\nu}, \quad B_{\mu\nu} = B^{\rho\gamma} g_{\rho\mu} g_{\gamma\nu}$$

~~there~~ Inner product

$$A^{\mu\nu} B_{\mu\nu} = A_{\alpha\beta} B^{\rho\gamma} g^{\alpha\mu} g_{\rho\mu} g^{\beta\nu} g_{\gamma\nu}$$

$$= A_{\alpha\beta} B^{\rho\gamma} \delta^{\alpha}_{\rho} \delta^{\beta}_{\gamma}$$

$$= A_{\alpha\beta} B^{\alpha\gamma} \delta^{\beta}_{\gamma}$$

$$= A_{\alpha\beta} B^{\alpha\beta}$$

3

$A^{\mu}_{\nu} \to$ symmetric $A^{\mu}_{\nu} = A^{\nu}_{\mu} \to x^i$

$\bar{x}^i$

$$\bar{A}^i_j = \frac{\partial \bar{x}^i}{\partial x^\mu} \frac{\partial x^\nu}{\partial \bar{x}^j} A^\mu_\nu$$

$\to$ interchanging $i, j, \mu, \nu \Rightarrow \bar{A}^j_i = \frac{\partial \bar{x}^j}{\partial x^\nu} \frac{\partial x^\mu}{\partial \bar{x}^i} A^\nu_\mu$

$$= \frac{\partial \bar{x}^j}{\partial x^\nu} \frac{\partial x^\mu}{\partial \bar{x}^i} A^\mu_\nu$$

$$\bar{A}^i_j \neq \bar{A}^j_i$$

5

$C^{ij} A_i A_j \to$ is scalar

$$\Rightarrow \bar{C}^{ij} \bar{A}_i \bar{A}_j = C^{pq} A_p A_q$$

$$\bar{A}_i = \frac{\partial x^p}{\partial \bar{x}^i} A_p \qquad \bar{B}_j = \frac{\partial x^q}{\partial \bar{x}^j} B_q$$

$$\left[\bar{C}^{ij} \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} - C^{pq} \right] A_p B_q = 0$$

$\Rightarrow C^{pq} = \bar{C}^{ij} \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} \to$ shows that C^{ij} transforms like 2nd rank Tensor

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$$A_j = g_{jk} A^k, \quad A^k = g^{jk} A_j$$

$$g^{jk} \quad g_{jk}$$

$$A_j = g_{jk} A^k \Rightarrow A_j A^k = g_{jl} A^l g^{mk} A_m$$

$$\Rightarrow A_j A^j = g_{jl} g^{mj} A^l A_m$$

$$\Rightarrow g_{jl} g^{mj} = \delta^m_l$$

$$ds^2 = g_{ij}\, dx^i dx^j = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 \quad [x^0 = ct, \; x^1 = x \quad x^2 = y$$

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

7

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} -1 & & 0 \\ & -1 & \\ 0 & & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\vec{r} \to -\vec{r}$$

$$\vec{A} \to (-\vec{A}) \quad \vec{B} \to (-\vec{B})$$

$$\vec{A} \times \vec{B} = (-\vec{A}) \times (-\vec{B}) = \vec{A} \times \vec{B}$$

$\to$ pseudovector

13

$$S \quad S' \quad S''$$

15

$$m_0 c^2 = \frac{m_{01} c^2}{\sqrt{1 - v_1^2/c^2}} + \frac{m_{02} c^2}{\sqrt{1 - v_2^2/c^2}} > m_{01} c^2 + m_{02} c^2$$

$$\Rightarrow m_0 > (m_{01} + m_{02})$$

17

18

$$a_\mu v^\mu = 0$$

$$v_\mu v^\mu = \text{L.I} \left(-c^2\right)$$

$$\frac{d}{d\tau} \left(v_\mu v^\mu\right) = 0$$

$$v_\mu \frac{d v^\mu}{d\tau} + v^\mu \frac{d v_\mu}{d\tau} = 0$$

$$v_\mu a^\mu + v^\mu a_\mu = 0$$

$$v \cdot a = 0$$