Step Potential:-

Consider a particle of mass m and energy E moving along x-axis from left to right towards a region having a potential step as shown in Fig-1. described by

$$V(x) = \begin{cases} 0 & \text{for } -\infty < x < 0 \\ V_0 & \text{for } 0 < x < \infty \end{cases}$$

we assume that $E < V_0$. The motion of the particle is governed by time-independent Schrödinger equation

$$\frac{d^2\psi(x)}{dx^2} + \frac{2m}{\hbar^2}\left[E - V(x)\right]\psi(x) = 0 \quad \cdots (1)$$

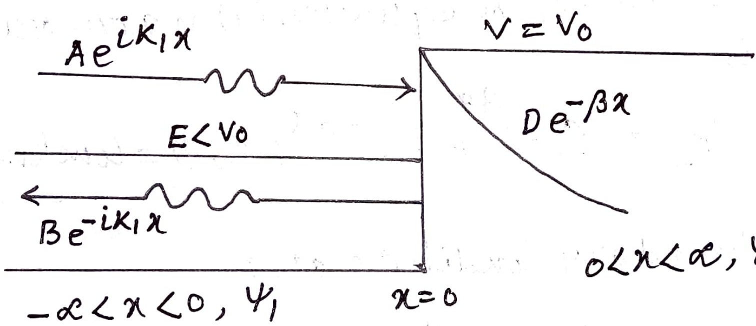

Fig-1.

Region $-\infty < x < 0$: $\psi = \psi_1$, $V = 0$ $(E < V_0)$:

Schrödiner equation for region $-\infty < x < 0$ is

$$\frac{d^2\psi_1}{dx^2} + \frac{2mE}{\hbar^2}\psi_1 = 0 \Rightarrow \frac{d^2\psi_1}{dx^2} + k_1^2\psi_1 = 0; \quad k_1^2 = \frac{2mE}{\hbar^2} \quad \cdots (2)$$

The equation (2) is a simple harmonic equation having oscillatory solution given by

$$\psi_1(x) = Ae^{ik_1x} + Be^{-ik_1x} = \psi_{in} + \psi_{ref}; \quad A, B = \text{Constants} \quad \cdots (3)$$

k_1 is a real quantity.

Region $0<x<\infty$: $\psi=\psi_2$ and $V=V_0$

Schrödinger equation for region $0<x<\infty$ is

$$\frac{d^2\psi_2}{dx^2}+\frac{2m}{\hbar^2}(E-V_0)\psi_2=0$$

$$\Rightarrow \frac{d^2\psi_2}{dx^2}-\frac{2m}{\hbar^2}(V_0-E)\psi_2=0$$

$$\Rightarrow \frac{d^2\psi_2}{dx^2}-\beta^2\psi_2=0;\ \beta^2=\frac{2m}{\hbar^2}(V_0-E) \quad \cdots (4)$$

β is a real quantity as $E<V_0$:

The solution of the equation (4) is non-oscillatory and is

$$\psi_2 = Ce^{\beta x}+De^{-\beta x};\ C, D = \text{Constants} \quad \cdots (5)$$

The boundary conditions are :

(i) $\psi=0$ at $x=\pm\infty$

(ii) $\psi=$ Continuous at the boundary (at $x=0$)

(iii) $\frac{d\psi}{dx}=$ Continuous at the boundary (at $x=0$)

Imposing boundary condition (i) on equation (5) we get

$$0 = Ce^{\beta(\infty)}+De^{-\beta(\infty)} \quad [\because x=+\infty]$$

$$= Ce^{\infty}$$

To prevent R.H.S of eqn (5) from diverging we choose $C=0$

Hence equation (5) becomes

$$\psi_2 = D e^{-\beta x} \text{ for } 0 < x < \alpha \quad \cdots\cdots (6)$$

Hence the solutions for wavefunctions for $E < V_0$ are

$$\left.\begin{aligned} \psi_1 &= A e^{iK_1 x} + B e^{-iK_1 x} = \psi_{in} + \psi_{ref} \; (-\alpha < x < 0) \\ \psi_2 &= D e^{-\beta x} \; (0 < x < \alpha) \end{aligned}\right\} \cdots (7)$$

A, B and D are determined from the boundary conditions (ii) and (iii) and they are found to be

$$\left.\begin{aligned} B &= \frac{K_1 - i\beta}{K_1 + i\beta} A \\ \text{and } D &= \frac{2K_1}{K_1 + i\beta} A \end{aligned}\right\} \quad --- (8)$$

The position probability current densities of ψ_{in}, ψ_{ref} and ψ_2 are given by (using eqn 12).

$$J_{in} = \frac{\hbar K_1}{m} |A|^2, \quad J_{ref} = -\frac{\hbar K_1}{m} |A|^2 \text{ and}$$

$$J_{transmitted} = 0 \text{ for } \psi_2$$

Reflection Coefficient represents the probability of reflection and is given by

$$R = \left| \frac{J_{ref}}{J_{in}} \right| = \left| \frac{-\frac{\hbar K_1}{m} |A|^2}{\frac{\hbar K_1}{m} |A|^2} \right| = 1.$$

Transmission Coefficient is

$$T = \left| \frac{J_{transmission}}{J_{in}} \right| = \left| \frac{0}{\frac{\hbar K_1}{m} |A|^2} \right| = 0.$$

$\therefore R+T = 1+0 = 1$

$\therefore \boxed{R+T=1}$ This is the expression of Conservation of probability.

(Fig-2)

We consider the energy of the particle $E > V_0$:

Schrödinger equation in region $0<x<\infty$ is

$$\frac{d^2\psi_2}{dx^2} + \frac{2m}{\hbar^2}(E-V_0)\psi_2 = 0$$

$$\Rightarrow \frac{d^2\psi_2}{dx^2} + K_2^2\psi_2 = 0 \; ; \; K_2^2 = \frac{2m}{\hbar^2}(E-V_0) \quad \cdots (9)$$

$K_2 =$ real quantity

The solution is oscillator and does not have any reflected part as the wave travels along +ve x-direction and is given by

$$\psi_2 = \psi_{transmitted} = G e^{iK_2 x} \; ; \; G = \text{Constant} \quad --(10)$$

Hence the solution for wave function for $E>V_0$ is

$$\left.\begin{aligned} \psi_1 &= A e^{iK_1 x} + B e^{-iK_1 x} \; ; \; -\infty<x<0 \\ \psi_2 &= G e^{iK_2 x} \; ; \; 0<x<\infty \end{aligned}\right\} \quad \cdots (11)$$

Applying the boundary conditions (ii) and (iii) on $\psi_1, \psi_2, d\psi_1/dx$ and $\frac{d\psi_2}{dx}$, the constants are found to be

$$B = \frac{K_1-K_2}{K_1+K_2} A \text{ and } G = \frac{2K_1}{K_1+K_2} A$$

The position probability current density is given by

$$J = \frac{\hbar}{2im}\left(\psi^* \frac{d\psi}{dx} - \psi \frac{d\psi^*}{dx}\right) \quad \text{---(12)}$$

Using equation (12) we have

$$J_{in} = \frac{\hbar K}{m}|A|^2 \quad \text{---(13)}$$

$$J_{ref} = -\frac{\hbar K_1}{m}\left|\frac{K_1 - K_2}{K_1 + K_2}\right|^2 |A|^2 \quad \text{---(14)}$$

and $$J_{tr} = \frac{4\hbar K_1^2 K_2}{m(K_1+K_2)^2}|A|^2 \quad \text{---(15)}$$

Reflection Coefficient is defined as

$$R = \left|\frac{J_{ref}}{J_{in}}\right| = \frac{\left|-\frac{\hbar K_1}{m}\left|\frac{K_1-K_2}{K_1+K_2}\right|^2 |A|^2\right|}{\left|\frac{\hbar K_1}{m}|A|^2\right|}$$

$$= \left|\frac{K_1 - K_2}{K_1 + K_2}\right|^2$$

$$= \left|\frac{K_1\left(1 - \frac{K_2}{K_1}\right)}{K_1\left(1 + \frac{K_2}{K_1}\right)}\right|^2 = \left|\frac{1 - \frac{K_2}{K_1}}{1 + \frac{K_2}{K_1}}\right|^2$$

Now $$\frac{K_2}{K_1} = \frac{\sqrt{\frac{2m}{\hbar^2}(E - V_0)}}{\sqrt{\frac{2m}{\hbar^2}E}} = \sqrt{\frac{E - V_0}{E}} = \sqrt{1 - \frac{V_0}{E}}$$

$$\therefore R = \left|\frac{1 - \sqrt{1 - \frac{V_0}{E}}}{1 + \sqrt{1 - \frac{V_0}{E}}}\right|^2 \quad \text{---(16)}$$

Transmission Coefficient is defined as

$$T = \left| \frac{J_{tr}}{J_{in}} \right| = \frac{\left| \frac{4\hbar K_1^2 K_2}{m(K_1+K_2)^2} |A|^2 \right|}{\left| \frac{\hbar K_1}{m} |A|^2 \right|}$$

$$= \frac{4 K_1 K_2}{(K_1+K_2)^2}$$

$$= \frac{4\sqrt{\frac{2m}{\hbar^2}E} \cdot \sqrt{\frac{2m}{\hbar^2}(E-V_0)}}{\left(\sqrt{\frac{2m}{\hbar^2}E} + \sqrt{\frac{2m}{\hbar^2}(E-V_0)}\right)^2}$$

$$= \frac{4\sqrt{E(E-V_0)}}{(\sqrt{E}+\sqrt{E-V_0})^2}$$

$$= \frac{4\sqrt{E^2\left(1-\frac{V_0}{E}\right)}}{\left[\sqrt{E}\left(1+\sqrt{\frac{E-V_0}{E}}\right)\right]^2}$$

$$T = \frac{4\sqrt{1-V_0/E}}{\left(1+\sqrt{1-V_0/E}\right)^2} \quad \text{---} (17)$$

$$\therefore R+T = \frac{\left(1-\sqrt{1-\frac{V_0}{E}}\right)^2}{\left(1+\sqrt{1-V_0/E}\right)^2} + \frac{4\sqrt{1-V_0/E}}{\left(1+\sqrt{1-V_0/E}\right)^2}$$

$$= 1.$$

$$\therefore \boxed{R+T=1}$$

The eq$^{\underline{n}}$ (1) represents conservation of probability.

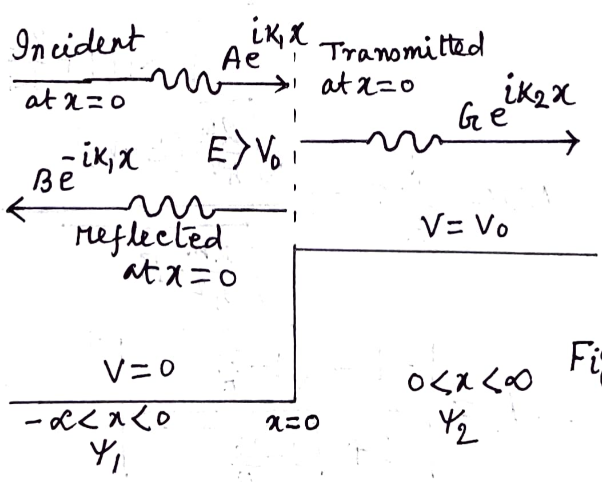

Fig-2