Plane Progressive Wave

Mathematical representation of a plane progressive wave :-

The function $f(vt-x)$ represents plane progressive wave propagating with phase velocity v in the positive x-direction. The function $f(vt-x)$ has the following characteristics

1) At a particular instant of time t, the quantity $(vt-x)$ and hence $f(vt-x)$ remain same at all points on a plane perpendicular to the x-axis. Thus the wave fronts are plane so that $f(vt-x)$ represents a plane wave propagating in the +ve x-direction. $(vt-x)$ is the phase of the wave.

2) At $x+dx$ and at $t+dt$

$$f[v(t+dt)-(x+dx)]$$

$$= f\left[vt + v\cancel{dt} - x - \frac{dx}{\cancel{dt}}dt\right]$$

$$= f(vt-x)$$

$\therefore$ The phase at (x,t) is same as the phase at $(x+dx, t+dt)$ where dx is the distance travelled in time dt.

Let $\psi(x,t) = f(vt-x)$

$$\therefore \frac{\partial\psi}{\partial t} = f'(vt-x)\cdot v$$

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 f''(vt-x)$$

Again $\frac{\partial \psi}{\partial x} = (-1) f'(vt-x)$

$$\frac{\partial^2 \psi}{\partial x^2} = f''(vt-x)$$

$$\therefore \frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial x^2} \text{ or } \boxed{\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}} \quad \text{---}(1)$$

This is one dimensional differential equation of plane progressive wave. Therefore a wave travelling in the +ve x-axis its equation of motion can be written as

$$\psi = f(vt-x) = f(vt - \vec{r}\cdot\hat{i})$$

For a wave travelling in any arbitrary direction $\vec{a}$ its equation of motion can be written as

$\psi(\vec{r},t) = f(vt - \vec{r}\cdot\hat{a})$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$
and $\hat{a} = l\hat{i} + m\hat{j} + n\hat{k}$
is unit vector along $\vec{a}$.

$$= f[vt - (lx+my+nz)] \quad \text{---}(2)$$

For a fixed time t, $\psi(\vec{r},t)$ will be same at all points on the plane $lx+my+nz = \text{Constant}$ where l, m, n are the d.c.s of the direction of propagation vector $\vec{a}$. In other words each plane perpendicular to the direction of propagation defines the locus of constant

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phase, known as waveform.

From (2)

$$\frac{\partial \psi}{\partial t} = v f'[vt-(lx+my+nz)]$$

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 f''[vt-(lx+my+nz)]$$

Again $$\frac{\partial^2 \psi}{\partial x^2} = l^2 f''[vt-(lx+my+nz)]$$

$$\frac{\partial^2 \psi}{\partial y^2} = m^2 f''[vt-(lx+my+nz)]$$

$$\frac{\partial^2 \psi}{\partial z^2} = n^2 f''[vt-(lx+my+nz)]$$

$$\therefore \frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}+\frac{\partial^2 \psi}{\partial z^2} = (l^2+m^2+n^2) f''[vt-(lx+my+nz)]$$

$$= f''[vt-(lx+my+nz)]$$

$$= \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$$

$$\Rightarrow \boxed{v^2 \nabla^2 \psi = \frac{\partial^2 \psi}{\partial t^2}} \text{-----} (3)$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$

The equation (3) is the 3-dimensional differential equation of plane progressive wave.

Equation of Plane progressive wave :-

Let a particle execute S.H.M about the mean position O and its equation is given by $y = a\sin\omega t$ (Fig-1).

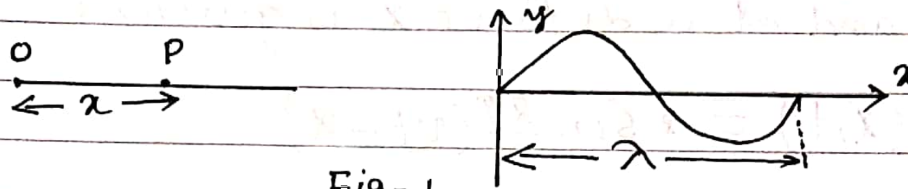

Fig-1.

To deduce the equation of plane progressive wave travelling in the +ve x-direction we have to find out the value of displacement at a distance x at time t from O. After a distance λ phase decreases by 2π. Therefore after a distance x phase decreases by $\frac{2\pi}{\lambda}.x$. Thus the phase of the wave at time t after travelling a distance x will be $(\omega t - \frac{2\pi}{\lambda}.x)$ and the displacement at P at time t will be

$$y = a\sin\left(\omega t - \frac{2\pi}{\lambda}x\right)$$

$$= a\sin\frac{2\pi}{\lambda}\left(\frac{2\pi}{T}.\frac{\lambda t}{2\pi} - x\right)$$

$$= a\sin\frac{2\pi}{\lambda}\left(\frac{\lambda}{T}t - x\right)$$

$$= a\sin\frac{2\pi}{\lambda}(vt - x) \quad \text{where } v = \lambda/T$$

$$\therefore y(x,t) = a\sin\frac{2\pi}{\lambda}(vt - x) \quad \text{- - - -(1)}$$

is the equation of a plane progressive wave travelling along +ve x-axis.

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Differential Equation of motion of a plane progressive wave :

The displacement of the particle at any instant of time and at a distance x is given by

$$y(x,t) = a \sin \frac{2\pi}{\lambda}(vt - x)$$

$$\therefore \frac{\partial y}{\partial t} = a \frac{2\pi}{\lambda} v \cos \frac{2\pi}{\lambda}(vt - x)$$

and $$\frac{\partial^2 y}{\partial t^2} = -v^2 \left(\frac{2\pi}{\lambda}\right)^2 a \sin \frac{2\pi}{\lambda}(vt - x) \quad ---(1)$$

Again $$\frac{\partial y}{\partial x} = -a\left(\frac{2\pi}{\lambda}\right) \cos \frac{2\pi}{\lambda}(vt - x)$$

and $$\frac{\partial^2 y}{\partial x^2} = -a\left(\frac{2\pi}{\lambda}\right)^2 \sin \frac{2\pi}{\lambda}(vt - x) \quad ---(2)$$

From (1) and (2)

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \Rightarrow \boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}} \quad ---(3)$$

The eqn (3) is the required differential equation of a plane progressive wave.

Problem 6: Prove that $y = e^{K(x - vt)}$ is a plane progressive wave.

Sol^{n}: $y = e^{K(x-vt)}$

$$\therefore \frac{\partial y}{\partial x} = K e^{K(x-vt)}$$

$$\frac{\partial^2 y}{\partial x^2} = K^2 e^{K(x-vt)} = K^2 y \quad \text{---} (1)$$

Again $$\frac{\partial y}{\partial t} = -Kv e^{K(x-vt)}$$

and $$\frac{\partial^2 y}{\partial t^2} = K^2 v^2 e^{K(x-vt)} = K^2 v^2 y \quad \text{--} (2)$$

From (1) and (2) $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$ This equation is a differential equation of a plane progressive wave. Thus $y = e^{K(x-vt)}$ is a equation of a plane progressive wave.

Relation between particle velocity and phase velocity:

The displacement of the particle at any instant of time t and at a distance x from the source of plane progressive wave is given by

$$y(x,t) = a \sin \frac{2\pi}{\lambda}(vt - x)$$

where v is the phase velocity and λ be its wavelength.

The particle velocity $u = \frac{\partial y}{\partial t} = \frac{2\pi}{\lambda} v \cdot a \cos \frac{2\pi}{\lambda}(vt - x)$ --- (1)

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Again
$$\frac{\partial y}{\partial x} = \text{Slope of The displacement curve}$$
$$= -\left(\frac{2\pi}{\lambda}\right) a \cos \frac{2\pi}{\lambda}(vt - x)$$

$$\therefore \frac{\partial y/\partial t}{\partial y/\partial x} = -v \Rightarrow \frac{\partial y}{\partial t} = -v\frac{\partial y}{\partial x}$$

$$\Rightarrow \boxed{u = -v\frac{\partial y}{\partial x}}$$

Spherical wave Solution :

Three dimensional wave equation is given by
$$\partial^2 \psi(\vec{r}, t) = v^2 \nabla^2 \psi(\vec{r}, t) \quad ----(1)$$

where $\psi = \psi(r, \theta, \phi, t)$ in spherical polar co-ordinate (r, θ, ϕ). If The wave have spherical symmetry Then ψ will be independent of θ and ϕ of spherical polar Coordinate system. In This Case The equation (1) becomes
$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right)$$

Putting $\psi = \psi'/r$ we get
$$\frac{\partial^2}{\partial t^2}\left(\frac{\psi'}{r}\right) = v^2 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\left(\psi'/r\right)\right)$$

$$\Rightarrow \frac{1}{r}\frac{\partial^2 \psi'}{\partial t^2} = \frac{v^2}{r^2}\frac{\partial}{\partial r}\left[r\frac{\partial \psi'}{\partial r} - \frac{r^2}{r^2}\psi'\right]$$

$$\Rightarrow \frac{1}{r}\frac{\partial^2 \psi'}{\partial t^2} = \frac{v^2}{r^2}\left[\cancel{\frac{\partial \psi'}{\partial r}} + r\frac{\partial^2 \psi'}{\partial r^2} - \cancel{\frac{\partial \psi'}{\partial r}}\right]$$

$$= \frac{v^2}{r}\frac{\partial^2 \psi'}{\partial r^2}$$

$$\Rightarrow \frac{\partial^2 \psi'}{\partial t^2} = v^2\frac{\partial^2 \psi'}{\partial r^2} \quad ---(2)$$

This is the same form as one dimensional wave equation. Hence the general solution is therefore

$$\psi' = f_1(vt-r) + f_2(vt+r)$$

$$\Rightarrow \psi = \frac{\psi'}{r} = \frac{1}{r}f_1(vt-r) + \frac{1}{r}f_2(vt+r) \quad --(3)$$

The first term of eqn (3) represents a spherical wave diverging from the origin of a coordinate system with velocity v. The second term represents a similar wave converging to the origin.
A diverging harmonic spherical wave can be represented by

$$\psi(r,t) = \frac{A}{r}\cos(\omega t - Kr) + \frac{B}{r}\sin(\omega t - Kr)$$

This solution shows that for a spherical wave, the amplitude falls off as $\frac{1}{r}$ with distance where K is the wave number $= 2\pi/\lambda$.

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Phase velocity and Group velocity :

An ideal plane wave travelling along the +ve x-direction in a medium can be represented by the equation

$$y = a \sin \frac{2\pi}{\lambda}(vt - x) \quad \text{------} (1)$$

Where v is the wave velocity. Since $v = \lambda\nu$, $\omega = 2\pi\nu$ and $K = 2\pi/\lambda$ hence from eqn (1) becomes

$$y = a \sin(\omega t - Kx) \quad \text{--------} (2)$$

The phase velocity $v_p = \frac{\omega}{K} = \frac{2\pi\nu}{2\pi/\lambda} = \lambda\nu = v =$ Wave velo.

The phase velocity is defined as the velocity with which a plane wavefornt, a plane of constant phase, propagates.

$$\therefore \omega t - Kx = \text{Constant}$$

$$\Rightarrow \omega - K\frac{dx}{dt} = 0$$

$$\Rightarrow \frac{dx}{dt} = v_p = \omega/K \quad \text{------} (3)$$

A group of two or more waves of nearly same frequencies are superimposed on each other, the resultant wave will have different properties from those of the individual wave. The resultant wave represents the wave group of large amplitude. The velocity with which the maximum of the group travels is called group velocity. It is defined as

$$v_g = \frac{d\omega}{dK} \quad \text{----------} (4)$$

For non-dispersive medium, the resultant wave consisting of groups of waves also moves with the same velocity as that of the constituent wave. So in non-dispersive medium group velocity and phase velocity are same.
For dispersive medium (ω/k) is not constant thus the group velocity and phase velocity are different to each other.

Let us consider two waves of slightly different wave-length and phase velocity as

$$y_1 = a\sin\frac{2\pi}{\lambda}(vt-x) \text{ and } y_2 = a\sin\frac{2\pi}{\lambda+d\lambda}\left[(v+dv)t-x\right] \quad \text{----(5)}$$

$$\therefore y = y_1+y_2 = a\sin 2\pi\left\{\frac{v}{\lambda}t-\frac{x}{\lambda}\right\}+a\sin 2\pi\left[\frac{v+dv}{\lambda+d\lambda}t-\frac{x}{\lambda+d\lambda}\right]$$

$$= 2a\cos 2\pi\left[\frac{\left(\frac{v}{\lambda}t-\frac{x}{\lambda}\right)-\left(\frac{v+dv}{\lambda+d\lambda}t-\frac{x}{\lambda+d\lambda}\right)}{2}\right]$$

$$\times \sin 2\pi\left[\frac{\left(\frac{v}{\lambda}t-\frac{x}{\lambda}\right)+\left(\frac{v+dv}{\lambda+d\lambda}t-\frac{x}{\lambda+d\lambda}\right)}{2}\right]$$

$$= 2a\cos\pi\left[\left(\frac{v}{\lambda}-\frac{v+dv}{\lambda+d\lambda}\right)t-\left(\frac{1}{\lambda}-\frac{1}{\lambda+d\lambda}\right)x\right]$$

$$\times \sin\pi\left[\left(\frac{v}{\lambda}+\frac{v+dv}{\lambda+d\lambda}\right)t-\left(\frac{1}{\lambda}+\frac{1}{\lambda+d\lambda}\right)x\right]$$

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Now

$$\frac{v}{\lambda} - \frac{v+dv}{\lambda+d\lambda} = \frac{\cancel{v\lambda}+vd\lambda-\cancel{v\lambda}-\lambda dv}{\lambda(\lambda+d\lambda)}$$

$$\simeq \frac{vd\lambda-\lambda dv}{\lambda^2}$$

$$\frac{1}{\lambda} - \frac{1}{\lambda+d\lambda} = \frac{\cancel{\lambda}+d\lambda-\cancel{\lambda}}{\lambda(\lambda+d\lambda)} \simeq \frac{d\lambda}{\lambda^2}$$

$$\frac{v}{\lambda} + \frac{v+dv}{\lambda+d\lambda} = \frac{v\lambda+vd\lambda+v\lambda+\lambda dv}{\lambda(\lambda+d\lambda)} \simeq \frac{2v\lambda}{\lambda^2} = \frac{2v}{\lambda}$$

$$\frac{1}{\lambda} + \frac{1}{\lambda+d\lambda} = \frac{\lambda+d\lambda+\lambda}{\lambda(\lambda+d\lambda)} \simeq \frac{2\lambda}{\lambda^2} = \frac{2}{\lambda}$$

Neglecting λdv, $vd\lambda$, $\lambda d\lambda$ and $d\lambda$ we get

$$y = 2a\cos\left[\frac{\pi t}{\lambda^2}(vd\lambda-\lambda dv) - \frac{\pi x\, d\lambda}{\lambda^2}\right]$$
$$\times \sin\left[\frac{\pi 2vt}{\lambda} - \frac{2\pi x}{\lambda}\right]$$

$$= 2a\cos\frac{\pi}{\lambda^2}d\lambda\left[\left(v-\lambda\frac{dv}{d\lambda}\right)t-x\right]\sin\frac{2\pi}{\lambda}(vt-x)$$

$$= 2a\cos\frac{2\pi}{\left(\frac{2\lambda^2}{d\lambda}\right)}\left[\left(v-\lambda\frac{dv}{d\lambda}\right)t-x\right]\sin\frac{2\pi}{\lambda}(vt-x) \quad \text{---(6)}$$

The resultant wave contains sine and cosine parts

The sine part denotes the velocity with which the wave advances and the cosine part modulates the amplitude and the velocity of the amplitude is $(v - \lambda \, dv/d\lambda)$ while the corresponding wavelength $(2\lambda^2/d\lambda)$. The quantity $(v - \lambda \, dv/d\lambda)$ is known as group velocity v_g of the wave i.e

$$\boxed{v_g = v - \lambda \frac{dv}{d\lambda}} \quad \text{------} (7)$$

for nondispersive medium $dv/d\lambda = 0$ and $\boxed{v_g = v}$ i.e group velocity = phase velocity

Normally $dv/d\lambda = +ve$ and $v_g < v$

The phase velocity $v = \omega/k$

Let us find

$$\frac{d\omega}{dK} = \frac{d(2\pi\nu)}{d(2\pi/\lambda)} = \frac{d\nu}{d(1/\lambda)} = \frac{d(v/\lambda)}{d(1/\lambda)}$$

$$= \frac{d(v/\lambda)}{-\frac{1}{\lambda^2} d\lambda}$$

$$= -\lambda^2 \frac{\left[\frac{1}{\lambda} dv - \frac{1}{\lambda^2} v \, d\lambda\right]}{d\lambda}$$

$$= -\lambda \frac{dv}{d\lambda} + v$$

$$= v - \lambda \frac{dv}{d\lambda} = v_g$$

$$\boxed{\therefore v_g = \frac{d\omega}{dK}}$$

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Problem 7: Prove that $v_g = \dfrac{c}{n + \omega \frac{dn}{d\omega}}$ where n = r.i of the medium, c = velo. of light in free space, ω = angular velocity of the wave.

Soln.: Phase velo. $v = \omega / K$ —— (1)

and group velo $v_g = \dfrac{d\omega}{dK}$ —— (2)

Now $n = n(\omega)$; n is the function of ω.

$$= \frac{c}{v} = \frac{cK}{\omega} \quad [\because v = \omega/K]$$

$$\therefore \frac{dn}{d\omega} = -\frac{cK}{\omega^2} + \frac{c}{\omega}\frac{dK}{d\omega}$$

$$= -\frac{n}{\omega} + \frac{cK}{\omega} \cdot \frac{1}{K}\frac{dK}{d\omega}$$

$$\therefore \frac{dn}{d\omega} = -\frac{n}{\omega} + \frac{n}{K}\frac{dK}{d\omega}$$

$$\Rightarrow \frac{n}{K}\frac{dK}{d\omega} = \frac{n}{\omega} + \frac{dn}{d\omega} = \frac{1}{\omega}\left[n + \omega\frac{dn}{d\omega}\right]$$

$$\Rightarrow \frac{n}{K} \cdot \frac{1}{v_g} = \frac{1}{\omega}\left[n + \omega\frac{dn}{d\omega}\right]$$

$$\Rightarrow v_g \frac{K}{n\omega} = \frac{1}{\left(n + \omega\frac{dn}{d\omega}\right)}$$

$$\Rightarrow v_g = \frac{\frac{n\omega}{K}}{\left(n + \omega\frac{dn}{d\omega}\right)} = \frac{c}{\left(n + \omega\frac{dn}{d\omega}\right)} \text{ (Proved)}$$

$$\therefore \boxed{v_g < c}$$

Problem 8: The angular frequency is 5 times of propagation constant. Calculate the phase and group velocities.

Soln: As per question $\omega = 5K$

$\therefore$ phase velocity $v = \omega/K = 5$ units.

$$v_g = v - \lambda \frac{dv}{d\lambda}$$

As $v =$ Constant $\frac{dv}{d\lambda} = 0 \quad \therefore v_g = v = 5$ units.

Problem 9: If the angular velocity ω and the propagation constant K of a wave are related by $\omega^2 = c^2K^2 + \omega_0^2$ where c and ω_0 are constants then show that the product of group velo. and phase velo. is a constant.

Soln: $\omega^2 = c^2K^2 + \omega_0^2$

Differentating w.r. to K we have

$$2\omega \frac{d\omega}{dK} = 2Kc^2$$

$$\Rightarrow \frac{d\omega}{dK} = \frac{c^2K}{\omega}$$

$\therefore$ group velo $v_g = \frac{d\omega}{dK} = \frac{c^2K}{\omega} = \frac{c^2}{\omega/K} = c^2/v$

$\therefore v_g v = c^2 =$ Constant (Proved).

Problem 10: Neglecting the effects of surface tension and finite depth, the wave velocity of water waves of wavelength λ is given by $v = \sqrt{\frac{g\lambda}{2\pi}}$.

Prove that the group velo. is half the wave velo. Calculate the group velo. and wave velo. for $\lambda = 1000$ meters

Sol^n: $v = \sqrt{\frac{g\lambda}{2\pi}} = \sqrt{g/K}$ as $K = 2\pi/\lambda$

$$\Rightarrow \frac{\omega}{K} = \sqrt{g/K} \Rightarrow \omega = \sqrt{gK} = (gK)^{1/2}$$

$$\therefore v_g = \frac{d\omega}{dK} = g^{1/2} \cdot \frac{1}{2} K^{-1/2} = \frac{1}{2}\sqrt{\frac{g}{K}} = \frac{v}{2}$$

$$\therefore \boxed{v_g = v/2}$$

Now $v = \sqrt{\frac{g\lambda}{2\pi}} = \sqrt{\frac{9.8 \times 1000}{2 \times 3.14}} = 39.5$ m/s

$$v_g = v/2 = \frac{39.5}{2} = 19.75 \text{ m/s.}$$

Problem 11: If the frequency of matter wave is $\frac{1}{2}\frac{mv^2}{h}$ and wavelength $\frac{h}{mv}$. Show that $v_g = v$.

Sol^n: Frequency $\nu = \frac{1}{h} \cdot \frac{1}{2} mv^2$; $\lambda = \frac{h}{mv}$

$$\therefore \omega = 2\pi\nu = \frac{\pi}{h} mv^2$$

$$\frac{d\omega}{dv} = \frac{2\pi}{h} mv$$

Now $K = 2\pi/\lambda = \frac{2\pi m v}{h}$

$\therefore \frac{dK}{dv} = \frac{2\pi m}{h}$

$\therefore v_g = \frac{d\omega}{dK} = \frac{d\omega/dv}{dK/dv} = \frac{2\pi}{h} m v \times \frac{h}{2\pi m}$

$= v$ (Proved)

$\boxed{\text{Problem 12:}}$ The phase velo. v of transverse wave in a crystal of atomic separation 'a' is given by

$$v = c\,\frac{\sin\left(\frac{1}{2}Ka\right)}{\left(\frac{1}{2}Ka\right)}$$

where K is the wave number and c is a constant. Show that

$$v_g = c\cos\left(\tfrac{1}{2}Ka\right)$$

Soln :- $v = c\,\frac{\sin\left(\frac{1}{2}Ka\right)}{\left(\frac{1}{2}Ka\right)}$

Now $\frac{1}{2}Ka = \frac{1}{2}\cdot\frac{2\pi}{\lambda}a = \frac{\pi a}{\lambda}$

$\therefore v = \frac{c\lambda}{\pi a}\sin\left(\frac{\pi a}{\lambda}\right)$

$\frac{dv}{d\lambda} = \frac{c}{\pi a}\sin\left(\frac{\pi a}{\lambda}\right) - \frac{c}{\lambda}\cos\left(\frac{\pi a}{\lambda}\right)$

$v_g = v - \lambda\frac{dv}{d\lambda} = \frac{c\lambda}{\pi a}\cancel{\sin}\left(\frac{\pi a}{\lambda}\right) - \frac{c\lambda}{\pi a}\cancel{\sin}\left(\frac{\pi a}{\lambda}\right)$

$+ c\cos\left(\frac{\pi a}{\lambda}\right)$

$= c\cos\left(\frac{\pi a}{\lambda}\right) = c\cos\left(\frac{1}{2}Ka\right)$. Proved.

Time Average of K.E and P.E of a Longitudinal plane progressive wave (Sound wave):-

Let us consider a cylindrical section of the medium of cross-sectional area α. In this section let us consider a volume confined between the transverse sections AB and A′B′ where AB and A′B′ have coordinates x and $x+dx$ w.r. to some arbitrary origin O on the left as shown in fig-1.

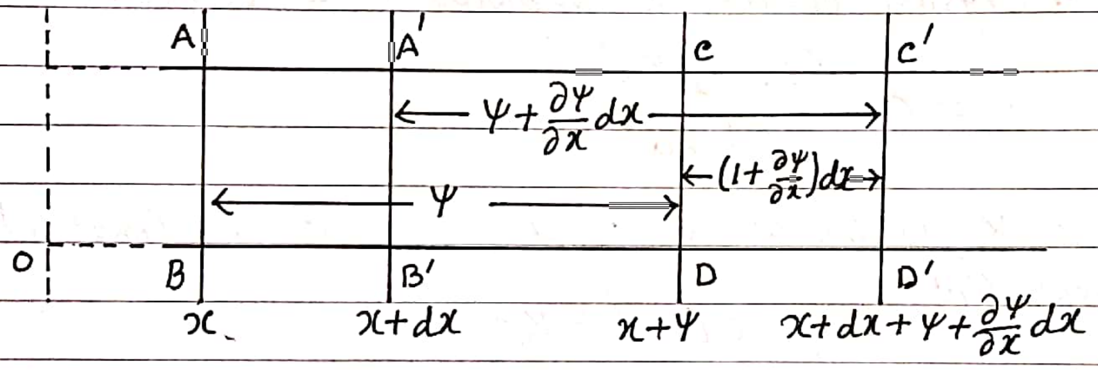

Fig-1

Let a plane progressive wave travels from left to right i.e in the +ve x-direction and the equation of the wave can be written as

$$\psi(x,t) = a\sin\frac{2\pi}{\lambda}(vt-x) \quad ------(1)$$

where a, λ and v are the amplitude, wavelength and wave velocity of the wave respectively.

The volume of the elementary volume ABA′B′ $= \alpha\, dx$

The K.E of this volume element at time t

$$= \frac{1}{2}\alpha\, dx\, \rho\left(\frac{\partial\psi}{\partial t}\right)^2 \quad \text{where } \rho = \text{density of medium}$$

Now
$$\frac{\partial \psi}{\partial t} = a\frac{2\pi}{\lambda} v \cos\frac{2\pi}{\lambda}(vt-x)$$

∴ K.E of the volume element

$$E_K = \frac{1}{2}\alpha\, dx\, \rho\, a^2 \frac{4\pi^2}{\lambda^2} v^2 \cos^2\frac{2\pi}{\lambda}(vt-x)$$

$$= \alpha\, dx\, \rho\, a^2 \frac{2\pi^2}{\lambda^2} v^2 \cos^2\frac{2\pi}{\lambda}(vt-x)$$

The time average of K.E of the volume element $ABA'B'$

$$\langle E_K\rangle = \alpha\, dx\, \rho\, a^2 \frac{2\pi^2}{\lambda^2} v^2 \frac{1}{2T}\int_0^T 2\cos^2\frac{2\pi}{\lambda}(vt-x)\,dt$$

$$= \alpha\, dx\, \rho\, a^2 \frac{\pi^2}{\lambda^2}\frac{v^2}{T}\left[\underbrace{\int_0^T \cos\frac{4\pi}{\lambda}(vt-x)\,dt}_{0} + \int_0^T dt\right]$$

$$= \alpha\, dx\, \rho\, \frac{\pi^2 a^2}{\lambda^2}\frac{v^2}{T}\cdot T$$

$$= \frac{\alpha\, dx\, \rho\, \pi^2 a^2 v^2}{\lambda^2} \quad \text{------------(2)}$$

The section AB is displaced by ψ to take up the position CD while $A'B'$ is displaced by $\psi + \frac{\partial \psi}{\partial x}dx$ to take up the position $C'D'$. Clearly the co-ordinates of CD and $C'D'$ will be $x+\psi$ and $x+dx+\psi+\frac{\partial\psi}{\partial x}dx$ respectively

Therefore the volume element $= \alpha\, dx\left(1+\frac{\partial\psi}{\partial x}\right)$

∴ Change in volume element $= \alpha\, dx\left(1+\frac{\partial\psi}{\partial x}\right) - \alpha\, dx$
$$= \alpha\, dx\, \partial\psi/\partial x$$

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Let P be the pressure acting on the undisturbed state ABA'B' and P+dP is the pressure acting in the disturbed state CDC'D'. The average pressure acting on the volume element

$$= \frac{P+P+dP}{2} = P + \frac{1}{2}dP$$

Now K = Bulk modulus of elasticity of the medium

$$= \frac{\text{volume Stress}}{\text{volume Strain}} = -\frac{dP}{\frac{\alpha dx \, \partial\psi/\partial x}{\alpha dx}}$$

$$dP = -K\frac{\partial\psi}{\partial x}$$

$$\therefore \text{Work done } E_P = \left(P + \frac{1}{2}P\right)\left(-\alpha dx \frac{\partial\psi}{\partial x}\right)$$

$$= \left[P + \frac{1}{2}\left(-K\frac{\partial\psi}{\partial x}\right)\right]\left(-\alpha dx \frac{\partial\psi}{\partial x}\right)$$

$$= -\alpha \, dx \, P\frac{\partial\psi}{\partial x} + \frac{1}{2}K\alpha \, dx\left(\frac{\partial\psi}{\partial x}\right)^2$$

Now $\psi(x,t) = a\sin\frac{2\pi}{\lambda}(vt - x)$

$$\therefore \frac{\partial\psi}{\partial x} = -a\frac{2\pi}{\lambda}\cos\frac{2\pi}{\lambda}(vt-x)$$

$$\therefore \langle E_P\rangle = -P\alpha \, dx \frac{1}{T}\int_0^T \left(\frac{\partial\psi}{\partial x}\right)dt + \frac{1}{2}K\alpha \, dx \frac{1}{T}\int_0^T \left(\frac{\partial\psi}{\partial x}\right)^2 dt$$

$$= P\alpha dx \frac{1}{T}\int_0^T a\frac{2\pi}{\lambda}\cos\frac{2\pi}{\lambda}(vt-x)\,dt + \frac{1}{2}K\alpha \, dx\left(\frac{4\pi^2}{\lambda^2}\right)a^2\frac{1}{T}\int_0^T \cos^2\frac{2\pi}{\lambda}(vt-x)\,dt$$

$$\therefore \langle E_P \rangle = P\alpha\, dx \frac{2\pi}{\lambda} a \frac{1}{T}\int_0^T \cos\frac{2\pi}{\lambda}(vt-x)\,dt$$

$$+ K\alpha\, dx \frac{2\pi^2}{\lambda^2} a^2 \cdot \frac{1}{2T}\int_0^T 2\cos^2\frac{2\pi}{\lambda}(vt-x)\,dt$$

$$= K\alpha\, dx \frac{2\pi^2}{\lambda^2} a^2 \frac{1}{2T}\int_0^T \left[\cos\frac{4\pi}{\lambda}(vt-x)+1\right]dt$$

$$= K\alpha\, dx \frac{2\pi^2}{\lambda^2} a^2 \frac{1}{2T}\cdot T$$

$$= \frac{\pi^2 a^2 \alpha\, dx}{\lambda^2} K = \frac{\pi^2 a^2 \alpha\, dx\, \rho v^2}{\lambda^2} \quad \text{---} (3)$$

$$\because v = \sqrt{K/\rho}$$
$$\Rightarrow v^2 \rho = K$$

From the equations (2) and (3) we have

$$\langle E_K \rangle = \langle E_P \rangle$$

Total time average energy is

$$\langle E \rangle = \langle E_K \rangle + \langle E_P \rangle = \frac{2\pi^2 a^2}{\lambda^2}(\alpha\, dx\, \rho v^2)$$

The energy density $u = \frac{\langle E \rangle}{\text{volume}}$

$$= \frac{2\pi^2 a^2}{\lambda^2} \frac{\alpha\, dx\, \rho v^2}{\alpha\, dx}$$

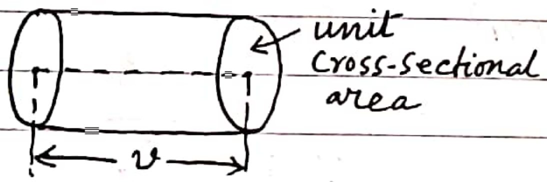
$$= \frac{2\pi^2 a^2}{\lambda^2} v^2 \rho = 2\pi^2 a^2 \nu^2 \rho \quad \text{---} (4)$$

where frequency $\nu = v/\lambda$.

The intensity of sound or acoustic intensity at a point is defined as the average rate of flow of energy through a unit area surrounding the point held normal to the direction of wave propagation.

$\therefore$ Intensity I = Energy contained within a cylinder of unit cross-section and length equal to the velocity of propagation

$$= u \times (1 \times v) = uv = \frac{2\pi^2 a^2}{\lambda^2} v^3 \rho$$



$$= 2\pi^2 a^2 \nu^2 v \rho \quad ---(5)$$

Relation between I and $(\partial P)_{max}$:-

The displacement of the particle at any instant of time at a distance x from the source of wave is given by

$$\psi(x,t) = a \sin \frac{2\pi}{\lambda}(vt - x) \quad ---(1)$$

and the Bulk modulus of elasticity of the medium is given by

$$K = -\frac{\partial P}{\left(\frac{\partial \psi}{\partial x}\right)} \quad \therefore \partial P = -K\left(\frac{\partial \psi}{\partial x}\right) \quad --(2)$$

From (1) and (2) we have

$$\partial P = \left(Ka\frac{2\pi}{\lambda}\right)\cos\frac{2\pi}{\lambda}(vt-x)$$

$$\Rightarrow \partial P = (\partial P)_{max}\cos\frac{2\pi}{\lambda}(vt-x) \quad ----(3)$$

where $(\partial P)_{max} = \frac{2\pi}{\lambda}Ka$ = pressure amplitude

$$\therefore a^2 = \frac{\lambda^2}{4\pi^2K^2}(\partial P)^2_{max} \quad ----(4)$$

Now

$$I = \frac{2\pi^2\rho v^3a^2}{\lambda^2} \Rightarrow \lambda^2 = \frac{2\pi^2\rho v^3a^2}{I} \quad --(5)$$

From (4) and (5)

$$a^2 = \frac{1}{4\pi^2K^2}\cdot\frac{2\pi^2\rho v^3a^2}{I}\cdot(\partial P)^2_{max}$$

$$\Rightarrow 1 = \frac{\rho v^3}{2v^4\rho^2}\frac{(\partial P)^2_{max}}{I}. \qquad \because v = \sqrt{K/\rho} \Rightarrow K^2 = v^2\rho$$

$$\Rightarrow I = \frac{1}{2v\rho}(\partial P)^2_{max} \quad ------(6)$$

Let $\partial P = P$ (say) and $(\partial P)_{max} = P_0$ (say). From (3)

$$P = P_0\cos\frac{2\pi}{\lambda}(vt-x)$$

Now $P^2_{r.m.s} = \frac{1}{T}\int_0^T P^2dt = \frac{1}{T}\cdot P_0^2\int_0^T \cos^2\frac{2\pi}{\lambda}(vt-x)dt$

$$= \frac{P_0^2}{2T}\cdot T = \frac{P_0^2}{2}$$

$$\therefore \boxed{I = \frac{P^2_{r.m.s}}{\rho v}} \quad ---(7)$$

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Velocity of Longitudinal wave in a Fluid :

Let a longitudinal wave is propagating along the +ve x-direction in a fluid (gas) medium through a uniform cylindrical column of cross-sectional area α. Let A and B be two right plane sections at a distance x and $x+dx$ respectively from left end, to be assumed as origin O. as shown in Fig-2. Thus the volume of the fluid between A and B is $\alpha\, dx$.

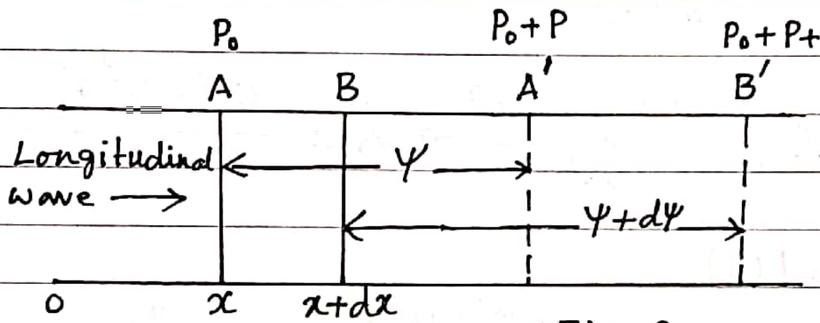

Fig-2

On the passage of the wave through the medium these plane sections will be displaced on account of the disturbances. If the new positions A' and B' are such that $AA' = \psi$ and $BB' = \psi + d\psi$

The increase in displacement between the plane sections is $\psi + d\psi - \psi = d\psi = \left(\frac{\partial \psi}{\partial x}\right) dx$

where $\frac{\partial \psi}{\partial x}$ = rate of change of displacement with distance.

The increase in volume between the plane sections

$$= \alpha\, d\psi = \alpha\, dx \left(\frac{\partial \psi}{\partial x}\right).$$

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If K be the bulk modulus of the medium then

$$K = -\frac{P}{\frac{\alpha\, dx\left(\frac{\partial \psi}{\partial x}\right)}{\alpha\, dx}} = -\frac{P}{\left(\frac{\partial \psi}{\partial x}\right)}$$

$$\therefore P = -K\left(\frac{\partial \psi}{\partial x}\right) \text{ at } A'$$

If $(\partial P/\partial x)$ be the pressure gradient then the pressure at B'

$$= P + \left(\frac{\partial P}{\partial x}\right) dx$$

$$= P - \frac{\partial}{\partial x}\left(K\frac{\partial \psi}{\partial x}\right) dx$$

$$= P - K\frac{\partial^2 \psi}{\partial x^2} dx$$

The pressure difference between B' and A'

$$= P - K\frac{\partial^2 \psi}{\partial x^2} dx - P = -K\frac{\partial^2 \psi}{\partial x^2} dx$$

and is directed from B' to A'

$\therefore$ Force acting on the slice $A'B'$ is

$$F = -K\left(\frac{\partial^2 \psi}{\partial x^2}\right) dx\, \alpha \quad = \text{Pressure} \times \text{area}$$

Thus from Newton's law of motion

$$F = -(\alpha\, dx\, \rho)\frac{\partial^2 \psi}{\partial t^2}$$

where ρ = density of the medium and $-\left(\frac{\partial^2 \psi}{\partial t^2}\right)$ is

The acceleration due to force

$$\therefore \; \alpha \rho \, dx \frac{\partial^2 \psi}{\partial t^2} = -K \alpha \, dx \frac{\partial^2 \psi}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial t^2} = \frac{K}{\rho} \frac{\partial^2 \psi}{\partial x^2} = v^2 \frac{\partial^2 \psi}{\partial x^2}$$

where $v = \sqrt{K/\rho}$ -------------(1)
is the velocity of sound wave in a fluid (gas) medium and is known as Newton's formula. Newton assumed that during propagation of sound wave through gas, the temperature of the medium remains constant i.e the process of propagation of sound wave is an isothermal process. Thus

$$PV = \text{Constant}$$

Differentiating we get

$$P\,dV + V\,dP = 0 \Rightarrow P = -\frac{dP}{\left(\frac{dV}{V}\right)} = K$$

From (1) $\quad v = \sqrt{\frac{P}{\rho}}$ ---------- (2)

At NTP i.e at 0°C or 273K, the normal pressure $P = 0.76 \times 1.36 \times 10^4 \times 9.81 \; N/m^2$; density of air $\rho = 1.293 \; kg/m^3$

$$\therefore v = \sqrt{\frac{0.76 \times 1.36 \times 10^4 \times 9.81}{1.293}} = 280 \; m/s.$$

which is less than the experimental result 332 m/s.

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Laplace corrected this difference by assuming the process of propagation of sound wave through air is not isothermal but it is adiabatic process and used the relation

$$PV^{\gamma} = \text{Constant}$$

Differentiating we get

$$P\gamma V^{\gamma-1} dV + V^{\gamma} dP = 0$$

$$\Rightarrow P\gamma V^{\gamma-1} dV = -V^{\gamma} dP$$

$$\Rightarrow \gamma P = -\frac{dP}{(dV/V)} = K$$

$$\therefore \; v = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{1.41} \times 280 \text{ m/s} \quad \text{as } \gamma = \frac{C_p}{C_v} = 1.41 \text{ for air}$$

$$= 331.6 \text{ m/s}$$

Velocity of Transverse Vibrations in a String:

Let us consider a string fixed between $x=0$ and $x=l$ and transverse waves in the y-direction be set up in the string

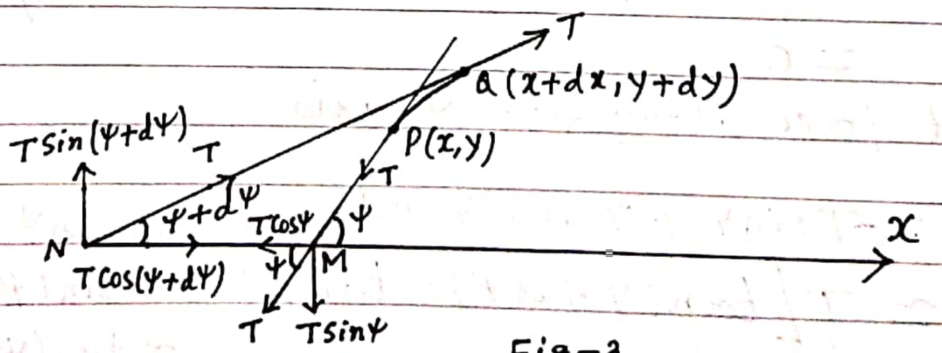


Fig-3.

As the string is perfectly ideal, T will be same throughout the string and acts tangentially at every point on the string.
Let PQ be an element of length ds of the string in the displaced position where the coordinates of P and Q are (x, y) and $(x+dx, y+dy)$ respectively as shown in fig-3.
The tension T at P acts along the tangent PM where PM makes an angle ψ with +ve x-axis.
Again at Q the tangent QN makes an angle $(\psi+d\psi)$ with +ve x-axis.

The components of T along $PM = T\cos\psi$ along $-$ve x-axis
and T along $PM = T\sin\psi$ along $-$ve y-axis

T along $NQ = T\cos(\psi+d\psi)$ along +ve x-axis
and $T\sin(\psi+d\psi)$ along +ve y-axis

Therefore net force acting along +ve x-axis

$$= -T\cos\psi + T\cos(\psi+d\psi)$$
$$= -T\cos\psi + T[\cos\psi\cos(d\psi) - \sin\psi\sin(d\psi)]$$
$$= -\cancel{T\cos\psi} + \cancel{T\cos\psi} \quad \text{for small displacement}$$
$$= 0$$

Net force along +ve y-axis

$$= -T\sin\psi + T\sin(\psi+d\psi)$$
$$\simeq T[\tan(\psi+d\psi) - \tan\psi]$$

$\because \sin\psi \approx \tan\psi = \left(\frac{\partial y}{\partial x}\right)_x$ and $\sin(\psi+d\psi) \approx \tan(\psi+d\psi) = \left(\frac{\partial y}{\partial x}\right)_{x+dx}$

$$= T\left[\left(\frac{\partial y}{\partial x}\right)_{x+dx} - \left(\frac{\partial y}{\partial x}\right)_x\right]$$

By using Taylor series expansion and neglecting higher order terms due to smallness we get

$$= T\left[\left(\frac{\partial y}{\partial x}\right)_x + \left(\frac{\partial^2 y}{\partial x^2}\right)_x dx - \left(\frac{\partial y}{\partial x}\right)_x\right]$$

$$= T\frac{\partial^2 y}{\partial x^2}dx$$

The mass of the element $ds \simeq dx$ is $m\,dx$ where m is the mass of the string per unit length. From Newton's law of motion we have

$$T\frac{\partial^2 y}{\partial x^2}dx = m\,dx\frac{\partial^2 y}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\frac{\partial^2 y}{\partial x^2} = v^2\frac{\partial^2 y}{\partial x^2} \quad ---(1)$$

where $v = \sqrt{T/m}$ ---------(2)

The eqn (1) is the differential wave equation and v is the velocity of transverse waves along the string.

The equation (1) has the solution of the form

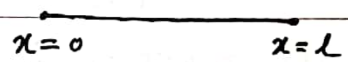
$$y = f_1(vt-x) + f_2(vt+x)$$

where $f_1(vt-x)$ represents a wave travelling in

The positive x-direction with velocity v and $f_2(vt+x)$ represents a wave travelling in the $-ve$ x-direction with velocity v. Since the string is fixed at $x=0$ and $x=l$, the transverse vibration be set up in the string, the wave formed be reflected back from the two ends and as such stationary waves will be formed.

Solution of the differential eqn (1) $\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$:-

The differential wave equation of transverse waves along the string fixed at both ends is given by



$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \quad \text{------} (1)$$

The displacement y is the function of x and t. The general solution of the wave eqn (1) may be obtained using the method of separation of variables. Let

$$y(x,t) = T(t)X(x) = TX \quad \text{---} (2)$$

Using (2) in (1) we get

$$\frac{\partial^2}{\partial t^2}(XT) = v^2 \frac{\partial^2}{\partial x^2}(XT)$$

$$\Rightarrow X \frac{\partial^2 T}{\partial t^2} = v^2 T \frac{\partial^2 X}{\partial x^2}$$

Dividing bothsides by XT we have

$$\frac{1}{v^2 T} \frac{d^2 T}{dt^2} = \frac{1}{X} \frac{d^2 X}{dx^2}$$

L.H.S of the above equation is the function t only while R.H.S is the function of x only. Therefore the equality implies that each side should be equal to a constant $-K^2$ (say).

i.e $\frac{1}{v^2 T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2} = -K^2$

$\therefore \frac{d^2T}{dt^2} = -(vK)^2 T = 0$ ---- (3)

The solution of eqn (3) is

$T(t) = A\sin(vK)t + B\cos(vK)t$ ---- (4)

Again

$\frac{d^2X}{dx^2} + K^2X = 0$ ---- (5)

The solution of eqn (5) is

$X(x) = C\sin Kx + D\cos Kx$ ------ (6)

where A, B, C and D are arbitrary constants and are determined from initial conditions.

The general solution of eqn (1) is

$y(x,t) = (A\sin(vK)t + B\cos(vK)t)(C\sin Kx + D\cos Kx)$ ---- (7)

At $x=0$; $y=0$ then from (7)

$0 = D[A\sin(vK)t + B\cos(vK)t]$

As $A\sin(vK)t + B\cos(vK)t \neq 0$ $\therefore$ $\boxed{D=0}$

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$$\therefore\ y(x,t) = C \sin kx\,[A \sin(vk)t + B \cos(vk)t]$$

$$= \sin kx\,[a \sin(vk)t + b \cos(vk)t]$$

Putting $AC = a$ and $BC = b$.

At $x = l$; $y = 0$

$$\therefore\ 0 = \sin kl\,[a \sin(vk)t + b \cos(vk)t]$$

As $a \sin(vk)t + b\cos(vk)t \neq 0$

$$\therefore\ \sin kl = 0 = \sin(n\pi) \quad \text{where } n = 1, 2, 3 \ldots$$

$$\therefore\ k = \frac{n\pi}{l} = k_n \text{ (say)}$$

and $\omega = vk = \dfrac{n\pi v}{l} = \omega_n$ (say).

Note that $B = 0$ and $n = 0$ are not physically acceptable because they make $y = 0$ for all t.

$$\therefore\ y_n = \sin\left(\frac{n\pi x}{l}\right)\left[a_n \sin\frac{n\pi v}{l}t + b_n \cos\frac{n\pi v}{l}t\right]$$

A particular solution for a particular value of n. Since the wave equation is linear and homogeneous the general solution is obtained by the principle of superposition as

$$y = \sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{l}x\right)\left[a_n \sin\frac{n\pi v}{l}t + b_n \cos\frac{n\pi v}{l}t\right] \quad ----(8)$$

Putting $a_n = A_n \cos\phi_n$ and $b_n = A_n \sin\phi_n$

where $A_n = \sqrt{a_n^2 + b_n^2}$ and $\phi_n = \tan^{-1}(b_n/a_n)$

The solution (8) can be written as

$$y = \sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{l}x \, \sin\left(\frac{n\pi v}{l}t + \phi_n\right)$$

$$= \sum_{n=1}^{\infty} A_n \sin k_n x \, \sin\left(\omega_n t + \phi_n\right) \quad \text{---(9)}$$

The solution (9) suggests that a stretched string of finite length and rigidly fixed at both ends can have only certain discrete frequencies. In the nth mode an element of the string vibrates simple harmonically with amplitude $A_n \sin k_n x$ and angular frequency ω_n. The frequency of vibration in the nth mode is

$$\nu_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\frac{n\pi v}{l} = \frac{nv}{2l} = \frac{n}{2l}\sqrt{\frac{T}{m}} \quad \text{---(10)}$$

The mode corresponding to $n=1$ is called fundamental mode of vibration and its frequency $\nu_1 = \frac{1}{2l}\sqrt{T/m}$ is called the fundamental or first harmonic frequency.

Energy of Vibrating String :-

The displacement $y = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{l}x \, \sin\left(\frac{n\pi v}{l}t + \phi_n\right)$

--- (1)

Let us consider an element of length dx at (x,y) at time t

$\therefore$ K.E of the element at time t is $= \frac{1}{2} m dx \left(\frac{\partial y}{\partial t}\right)^2$

$\therefore$ Total K.E of the string fixed at $x=0$ and $x=l$ is

$$K.E = \frac{1}{2}\int_0^l m dx \left(\frac{\partial y}{\partial t}\right)^2$$

$$= \frac{\pi^2 T}{4l}\sum_{n=1}^{\infty} A_n^2 n^2 \cos^2\left(\frac{n\pi v}{l} t + \phi_n\right) \text{---(2)}$$

The P.E of the string is

$$P.E = \frac{T}{2}\int_0^l \left(\frac{\partial y}{\partial x}\right)^2 dx$$

$$= \frac{\pi^2 T}{4l}\sum_{n=1}^{\infty} A_n^2 n^2 \sin^2\left(\frac{n\pi v}{l} t + \phi_n\right) \text{---(3)}$$

$\therefore$ Total energy $= K.E + P.E$ (from (2) & (3))

$$= \frac{\pi^2 T}{4l}\sum_{n=1}^{\infty} A_n^2 n^2 \text{------(4)}$$

Therefore the energy of vibration of nth mode is

$$E_n = \frac{\pi^2 T}{4l} A_n^2 n^2$$

$$= \frac{l}{4}\frac{\pi^2 T n^2}{l^2 m} A_n^2 \cdot m = \frac{lm}{4}\omega_n^2 A_n^2$$

$$= \frac{M}{4}\omega_n^2 A_n^2 \quad (\because lm = M \text{ total mass})$$

$$\boxed{\therefore E_n = \frac{M}{4}\omega_n^2 A_n^2} \text{----(5)}$$

Plucked String:

Let us consider a string fixed at $x=0$ and $x=l$, let the string be plucked transversely to its length $x=h$ by K and released, transverse vibrations will be set up in the string which on reflection from the ends $x=0$ and $x=l$ will produce stationary waves is given by

$$y=\sum_{s=1}^{\infty}\left(a_s \cos\frac{s\pi v}{l}t+b_s \sin\frac{s\pi v}{l}t\right)\sin\frac{s\pi x}{l} \quad ---(1)$$

The boundary conditions are

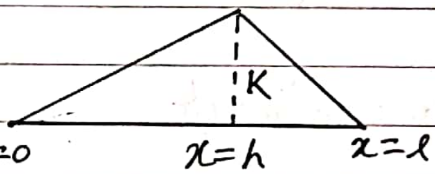


Fig-1

1) $\frac{\partial y}{\partial t}=0$ at $t=0$ for all points $x=0$

2) $y=K$ at $x=h$ at $t=0$

$$\therefore \frac{\partial y}{\partial t}=\sum_{s=1}^{\infty}\frac{s\pi v}{l}\left\{-a_s \sin\frac{s\pi v}{l}t+b_s\cos\frac{s\pi v}{l}t\right\}\sin\frac{s\pi x}{l}$$

$$\left.\frac{\partial y}{\partial t}\right)_{t=0}=0=\sum_{s=1}^{\infty}\frac{s\pi v}{l}\left\{b_s\right\}\sin\frac{s\pi x}{l}$$

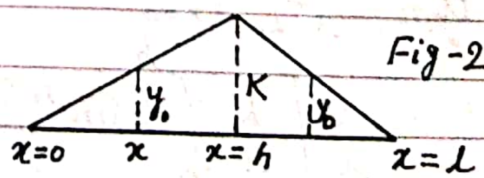
It implies $b_1=b_2=b_3=\cdots=0$ ------(2)

Therefore equation (1) becomes

$$y=\sum_{s=1}^{\infty}a_s\cos\frac{s\pi v}{l}t\cdot\sin\frac{s\pi x}{l} \quad ------(3)$$

Let $y=y_0$ at $t=0$ and at x then

Fig-2



$$y_0 = \sum_{s=1}^{\infty} a_s \sin \frac{s\pi}{l} x$$

Multiplying both sides by $\sin \frac{n\pi}{l} \cdot x$ and integrate from $x=0$ to $x=l$

i.e $$\int_0^l y_0 \sin \frac{n\pi}{l} x \, dx = \sum_{s=1}^{\infty} a_s \int_0^l \sin \frac{n\pi}{l} x \sin \frac{s\pi}{l} \cdot x \, dx$$

Since $$\int_0^l \sin \frac{n\pi}{l} x \sin \frac{s\pi}{l} x \, dx = 0 \text{ for } n \neq s$$
$$= \frac{l}{2} \text{ for } n = s$$

We can write

$$\int_0^l y_0 \sin \frac{n\pi}{l} x \, dx = a_n \, l/2$$

$$\therefore \quad \boxed{a_n = \frac{2}{l} \int_0^l y_0 \sin \frac{n\pi}{l} x \, dx} \quad \text{-----(4)}$$

From Fig-2 for $0 < x < h$:

$$\frac{K}{y_0} = \frac{h}{x} \Rightarrow y_0 = \frac{Kx}{h}$$

for $h < x < l$: $$\frac{K}{y_0} = \frac{l-h}{l-x} \Rightarrow y_0 = \frac{K(l-x)}{(l-h)}$$

$$a_n = \frac{2}{l}\left[\int_{x=0}^{x=h} \frac{Kx}{h} \sin \frac{n\pi}{l} x \, dx + \int_{x=h}^{x=l} \frac{K(l-x)}{(l-h)} \sin \frac{n\pi}{l} x \, dx\right]$$

$$= \frac{2}{l}\left[\frac{K}{h} \int_0^h x \sin \frac{n\pi x}{l} dx + \frac{K}{(l-h)} \int_h^l (l-x) \sin \frac{n\pi x}{l} dx\right]$$