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Conservation of Charge

We know that total electric charge is conserved in nature, i.e. if we generate or eliminate a positive charge, it will always accompanied by a negative charge of equal magnitude. If the total electric charge Q in any volume V changes with time, it must be attributed to a net flow of charge (i.e. an electric current) across the surface S of volume V.

We consider two different surfaces S_1 and S_2 bounded by a closed loop C such that a volume V is contained between the surfaces. Let $I_1 = \int \vec{J} \cdot d\bar{S}_1$ and $I_2 = \int \bar{J} \cdot d\bar{S}_2$ are the currents flowing through surfaces S_1 and S_2. Here I_1 enters volume V through S_1 and I_2 [redacted] exits volume V through S_2.

If $I_1 \neq I_2$, then the net charge (Q) contained in V will increase with time.

$\therefore$ We can write $\dfrac{dQ}{dt} = I_1 - I_2$

$$\text{or, } \frac{d}{dt}\int \rho \, dv = \int \bar{J} \cdot d\bar{S}_1 - \int \bar{J} \cdot d\bar{S}_2$$

$$\text{or, } \int \frac{d\rho}{dt} dv = -\oint \bar{J} \cdot d\bar{S}$$

[Here S is the surface enclosing volume V.]

Applying divergence theorem, we get

$$\int \frac{d\rho}{dt} dv = -\oint \vec{\nabla} \cdot \vec{J} \, dv$$

$\therefore \ \bar{\nabla}.\bar{J} + \frac{\partial \rho}{\partial t} = 0$

This is the equation of continuity. It is the mathematical form of principle of conservation of electric charge.

Maxwell's Equations

Before Maxwell, the laws of electricity and magnetism were expressed by the following four equations:

$\bar{\nabla}.\bar{E} = \rho/\epsilon_0$ (Gauss's law in electrostatics) --- (1)

$\bar{\nabla}.\bar{B} = 0$ (Gauss's law in magnetostatics) -- (2)

$\bar{\nabla}\times\bar{E} = -\frac{\partial \bar{B}}{\partial t}$ (faraday's Law) - - - - - - -- (3)

$\bar{\nabla}\times\bar{H} = \bar{J}$ (Ampere's law) - - - - - - - - (4)

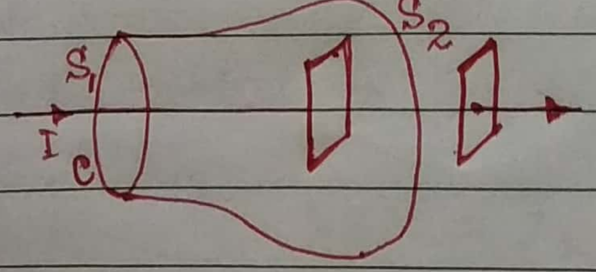
The Gauss's law for electrostatics tells us how the electric field behaves around electric charges. The Gauss's law for magnetostatics tells us that the magnetic field is divergenceless, i.e the lines of magnetic induction form closed loops. The third equation indicates that the time varying magnetic field induces electric field. The fourth equation tells that the source of current distribution produces magnetic field,

Faraday's law indicates that a time varying magnetic field gives rise to an electric field. From symmetry consideration we may expect that a time varying electric field can also produce a magnetic field. Infact Maxwell proved it to be true. Maxwell modified the previous field equations and removed the incompleteness of the interdependence of field equations. The equations thus obtained are known as the Maxwell's equations, which govern the behaviour of the classical electromagnetic field as we believe it today.

- Generalisation of Ampere's law and the concept of Displacement Current

Maxwell pointed out that equations (1–4) are not consistent. As the divergence of curl of any vector is always zero, by taking divergence of eq(4) we find $\bar{\nabla}.(\bar{\nabla}\times\bar{H}) = \bar{\nabla}.\bar{J} = 0$. For steady currents this is true. But for the time varying cases the equation of continuity is $\frac{\partial \rho}{\partial t} + \bar{\nabla}.\bar{J} = 0$, or, $\bar{\nabla}.\bar{J} = -\frac{\partial \rho}{\partial t}$. This indicates that $\bar{\nabla}.\bar{J}$ is not always zero.

Let us apply the Ampere's law $(\bar{\nabla}\times\bar{H} = \bar{J})$ to a circuit consisting of a parallel-plate capacitor.



Consider that the capacitor is charged by a current I. Consider a closed path C bounding the surfaces S_1 and S_2. The surface S_1 is intersecting a lead carrying current I and the surface S_2 is passing through the region between the capacitor plates.

Now, the Ampere's circuital law for the closed path C and surface S_1 gives

$$\oint_C \vec{H}\cdot d\vec{l} = \int_{S_1} \vec{J}\cdot d\vec{S} = I \quad \ldots (5)$$

and for the surface S_2 we have,

$$\oint_C \vec{H}\cdot d\vec{l} = \int_{S_2} \vec{J}\cdot d\vec{S} = 0 \quad \ldots (6)$$

(Since the current density is zero at every point on S_2.)

It is clear that eqs(5) and (6) are contradictory. Eq(5) is correct since it does not involve the capacitor and on the other hand eq(6) must be modified with the inclusion of the effect of the capacitor.

As electric charge is accumulated on the plate of the capacitor contained within volume enclosed by S_1 and S_2, the continuity equation is

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0 \quad \ldots (7)$$

where ρ is the charge density on the capacitor plate and is changing with time. So, some quantity must be added to the right hand side of the equation,

$$\vec{\nabla}\times\vec{H} = \vec{J},$$

that will be consistent with eqn(7).

We Know that the electric displacement $\vec{D}$ is related to the charge density ρ by

$$\vec{\nabla}\cdot\vec{D} = \rho \quad \cdots\cdots (8).$$

So, we have from eq(7)

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{D}) = \vec{\nabla}\cdot\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) = 0.$$

If we add $\frac{\partial \vec{D}}{\partial t}$ to the right-hand side of equation $\vec{\nabla}\times\vec{H} = \vec{J}$, then its divergence will satisfy eq(7), so the discrepency is removed. The electric field is almost confined between the plates of the capacitor, and $\frac{\partial \vec{D}}{\partial t}$ is negligible over S_1 and eq(5) remains unchanged. Now eq(6) becomes:

$$\oint_C \vec{H}\cdot d\vec{l} = \int_{S_2}\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)\cdot d\vec{S} = \int_{S_2}\frac{\partial \vec{D}}{\partial t}\cdot d\vec{S} = \oint_{S_1+S_2}\frac{\partial \vec{D}}{\partial t}\cdot d\vec{S}$$

$$\text{or, } \oint H\cdot d\vec{l} = \frac{d}{dt}\oint_{S_1+S_2}\vec{D}\cdot d\vec{S} = \frac{dq}{dt} = I.$$

(same as eq(5))

by Gauss's law in electrostatics, where 'q' is the charge in volume enclosed by S_1 and S_2 (i.e the charge on the capacitor plate).

With $\frac{\partial \vec{D}}{\partial t}$ included we have the generalised Amperes law (Ampere Maxwell's law).

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \cdots\cdots (9)$$

The quantity $\frac{\partial \vec{D}}{\partial t}$ is Known as the displacement current density. It was first introduced by Maxwell.

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Note that eq(9) makes $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{H}) = 0$ always.
The quantity $\frac{\partial \vec{D}}{\partial t}$ is a current in the sense that it can produce a magnetic field. As it is not linked with the motion of free charges, it has none of the other properties of current.

Now, the four fundamental equations of electromagnetism, known as the Maxwell's equations in SI units are —:

$$\vec{\nabla}\cdot\vec{D} = \rho \quad \text{-------} \quad (10)$$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \text{-------} \quad (11)$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{-----} \quad (12)$$

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \text{-------} \quad (13)$$

[Here, $\vec{D} = \epsilon\vec{E}$ and $\vec{B} = \mu\vec{H}$].

Here $\vec{E}$ and $\vec{D}$ are the electric field vectors, $\vec{B}$ and $\vec{H}$ are the magnetic field vectors, ρ is the free charge density and $\vec{J}$ is the free current density.

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Properties of Maxwell's Equations-

(i) Maxwell's equations are linear. It is directly related to the principle of superposition. So, if any two fields satisfy these equations then their sum will also satisfy the equations.

(ii) The form of these equations remains unchanged under Lorentz transformation. That is the Maxwell's equations are relativistic invariants.

(iii) Maxwell's equations are not symmetric with respect to electric and magnetic fields. This is due to the reason that electric charges exist in nature, while magnetic charges do not exist. These equations take up symmetric form in free space with $\rho = 0$ and $\vec{J} = 0$.

(iv) Maxwell's equations predict the existange of electromagnetic waves.

(v) Maxwell's equation include the equation of continuity.

Taking divergence of eq(13) we have

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{H}) = \vec{\nabla}\cdot\vec{J} + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{D})$$

or, $$0 = \vec{\nabla}\cdot\vec{J} + \partial\rho/\partial t \qquad \text{[using eq(10)]}$$

$$\therefore \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0.$$

This is the equation of continuity. So, we can say that the Maxwell's equations are consistent with the local conservation of charge.