Self Inductance

When a current flows in a circuit it produces a magnetic field in the sorrounding space. Hence a magnetic flux will be linked with the circuit. If the current through the circuit changes with time, then the flux will also change with time giving rise to an induced emf in the circuit. This phenomenon is known as the self-inductance. According to Lenz's law, the induced emf will oppose the change in current in the circuit. For this reason it is known as back emf.

The flux ϕ is proportional to the current I in the circuit.

$\therefore \phi = LI$, ---- (1) where L is the coefficient of self-inductance or simply self inductance of the circuit. It can be defined as the total flux linked with the circuit for unit current flowing in the circuit.

For a rigid stationary circuit the change in flux is caused by the change in current.

Thus $\dfrac{d\phi}{dt} = \dfrac{d\phi}{dI} \cdot \dfrac{dI}{dt}$ .... (2)

The induced emf in the circuit is given by

$$\varepsilon = -\frac{d\phi}{dt} = -L\frac{dI}{dt} \quad \text{[using eqn (1)]} \quad ---- (3)$$

$$\text{where, } L = \frac{d\phi}{dI} \quad ---- (4)$$

Eq(3) defines the self inductance as the emf induced in the circuit for a unit rate of change of current in it. If ϕ is not linearly related to I, the inductance defined by eq(4) is termed as the incremental inductance. The negative sign in eq(3) indicates that the induced emf opposes the flow of current. Thus the self inductance measures the ability of a circuit to oppose the variation of current in it. The SI units of ϕ and L are weber and henry respectively. The dimensions of ϕ and L are $[ML^2T^{-2}I^{-1}]$ and $[ML^2T^{-2}I^{-2}]$ respectively.

Mutual Inductance

Consider two closed circuits C_1 and C_2 placed close to each other. If I_1 is the current in C_1, then it will produce a magnetic field $\vec{B}_1$ at the position of C_2. Thus some flux of $\vec{B}_1$ will pass through C_2. If now I_1 is varied then the flux will also vary and according to laws of electro-magnetic induction there will be an emf induced in loop C_2. This phenomenon is known as mutual inductance.

The mutual inductance refers to the induction of an emf in a circuit due to change in current

in other circuits placed in its vicinity.
Let Φ_2 be the flux through C_2 due to a current I_1 in circuit C_1. In absence of ferromagnetic material Φ_2 is proportional to I_1.

$$\therefore \phi_2 = M_{21} I_1 \quad \cdots \cdots (1)$$

Similarly, the flux Φ_1 through C_1 due to a current I_2 in circuit C_2 will be

$$\Phi_1 = M_{12} I_2 \quad \cdots \cdots (2)$$

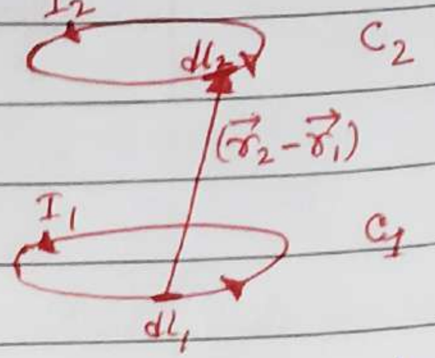
Here M_{12} and M_{21} are called the mutual inductance of the two circuits. (It will be shown that $M_{12} = M_{21}$).
Numerically, mutual inductance of two loops is equal to the flux linked with one loop due to unit current in other. The emf induced in loop C_2 due to current change in C_1 is,

$$\mathcal{E}_2 = -\frac{d\phi_2}{dt} = -\frac{d}{dt}(M_{21} I_1) = -M_{21}\frac{dI_1}{dt}$$

Similarly, $\mathcal{E}_1 = -\frac{d\phi_1}{dt} = -M_{12}\frac{dI_2}{dt}$

Thus, mutual inductance of two loops is numerically equal to the emf induced in one loop due to unit rate of change of current in other.

Neumann's Formula :-



Consider two loops C_1 and C_2 at rest. The loop C_1 carrying a current I_1 and produces a magnetic field $\bar{B}_1$ at the site of C_2. The flux linked with C_2 due to current in C_1 is,

$$\Phi_2 = \int_{S_2} \vec{B}_1 \cdot d\vec{S}_2 = \int_{S_2} (\bar{\nabla} \times \bar{A}_1) \cdot d\vec{S}_2 \quad \text{------ (1)}$$

where $\bar{A}_1$ is the magnetic vector potential corresponding to $\bar{B}_1$. Using Stoke's theorem.

$$\Phi_2 = \oint_{C_2} \bar{A}_1 \cdot d\bar{l}_2 \quad \text{------------ (2)}$$

Also, $\bar{A}_1$ can be expressed as,

$$\bar{A}_1 = \frac{\mu_0}{4\pi} \oint_{C_1} \frac{I_1 \, d\bar{l}_1}{|\bar{r}_2 - \bar{r}_1|} \quad \text{---------- (3)}$$

where $\bar{r}_1$ and $\bar{r}_2$ are the position vectors of the line elements $d\bar{l}_1$ and $d\bar{l}_2$ with respect to some arbitrary origin.

$$\therefore \Phi_2 = \frac{\mu_0 I_1}{4\pi} \oint_{C_2} \oint_{C_1} \frac{d\bar{l}_1 \cdot d\bar{l}_2}{|\bar{r}_2 - \bar{r}_1|} = M_{21} I_1$$

where, $$M_{21} = \frac{\mu_0}{4\pi} \oint_{C_2} \oint_{C_1} \frac{d\bar{l}_1 \cdot d\bar{l}_2}{|\bar{r}_2 - \bar{r}_1|} \quad \text{------ (4)}$$

Since $|\bar{r}_2 - \bar{r}_1| = |\bar{r}_1 - \bar{r}_2|$ and other integration may be interchanged we can write,

$$M_{21} = M_{12} = M = \frac{\mu_0}{4\pi} \oint_{C_1}\oint_{C_2} \frac{d\bar{l}_1 \cdot d\bar{l}_2}{|\bar{r}_1 - \bar{r}_2|} \quad \cdots (5)$$

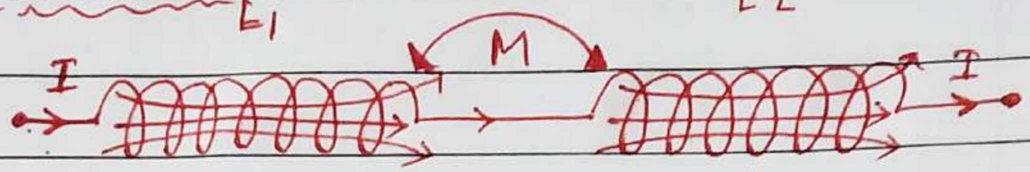
M is simply referred to as the mutual inductance between the loops. Eq(5) is called the Neumann's formula for the mutual inductance of two arbitrary loops.

Also, $\phi_1 = M_{12} I_2$ and $\phi_2 = M_{21} I_1$.
Now if $I_1 = I_2$, we can write $\phi_1 = \phi_2$
$[\because M_{12} = M_{21}]$.

So, the Neumann's formula shows that, "The flux linked with circuit 2 due to the current in circuit 1 is the same as the flux linked with circuit 1 due to the same current in circuit 2." This property is usually called the 'reciprocity theorem' for electromagnetic induction.

Inductance in series and parallel :-

Series connection -:



Consider two coils of self inductances L_1 and L_2 are connected in series. Let 'I' be the current through the coils and 'M' is the mutual inductance between the coils. For the figure above, the mutual flux between the coils will aid the self-flux. Here,

Flux linked with L_1 at time t is $(L_1 I + M I)$

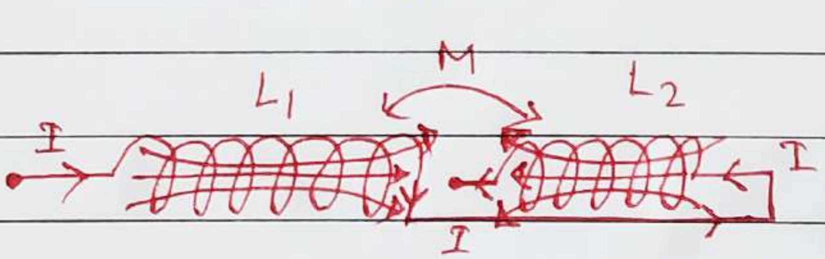
and " " " L_2 " " t " $(L_2 I + M I)$

$\therefore$ For both the coils the total flux is $(L_1 + L_2 + 2M) I$

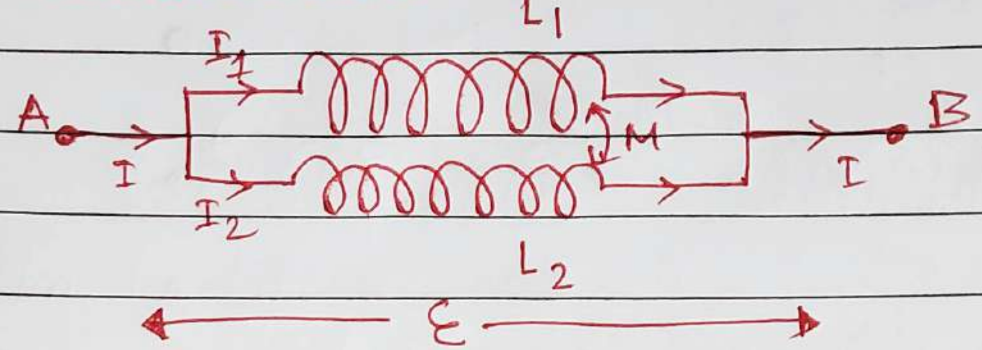
$\therefore$ Effective self inductance, i.e, the flux per unit current is, $L_{eff} = L_1 + L_2 + 2M$.

If the mutual flux oppose the self-flux linked with the coils, then the effective flux is

$$L'_{eff} = L_1 + L_2 - 2M.$$



Parallel connection-:



Consider two coils with self inductance L_1 and L_2 are connected in parallel and M be the mutual inductance between them. At time 't' the main current 'I' is divided into branch currents I_1 and I_2.

$$\therefore I = I_1 + I_2 \quad \text{or,} \quad \frac{dI}{dt} = \frac{dI_1}{dt} + \frac{dI_2}{dt} \quad \text{---- (1)}$$

Assuming the mutual flux aids the self-flux we can write, the effective flux linked with L_1 at time 't' as,

$$\phi_1 = L_1 I_1 + M I_2$$

and for 'L_2', $\phi_2 = L_2 I_2 + M I_1$

Now since the coils are in parallel the induced emf will be equal.

$$\therefore \varepsilon = -\frac{d\phi_1}{dt} = -\frac{d\phi_2}{dt}$$

$$\text{or,} \quad L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt} = -\varepsilon \quad \text{---- (2)}$$

$$\text{and} \quad L_2 \frac{dI_2}{dt} + M \frac{dI_1}{dt} = -\varepsilon \quad \text{------ (3)}$$

Solving we get, $\frac{dI_1}{dt} = \frac{-\varepsilon(L_2-M)}{(L_1L_2-M^2)}$

and $\frac{dI_2}{dt} = -\frac{\varepsilon(L_1-M)}{(L_1L_2-M^2)}$

From eq(1) we have,

$$\frac{dI}{dt} = \frac{-\varepsilon(L_1+L_2-2M)}{(L_1L_2-M^2)} \quad \cdots\cdots (4)$$

Now, if L_{eff} is the effective self-inductance, then we can write,

$$\varepsilon = -L_{eff}\frac{dI}{dt} \quad ------ (5)$$

From eq(4) and (5) we get

$$L_{eff} = \frac{L_1L_2-M^2}{L_1+L_2-2M} \quad ----- (6)$$

If the inductors are so connected that the mutual flux oppose the self flux then,

$$L'_{eff} = \frac{L_1L_2-M^2}{L_1+L_2+2M} \quad ----- (7)$$

If there is no magnetic coupling between the coils, then $M=0$, and

$$L_{eff} = \frac{L_1L_2}{L_1+L_2}$$

or, $\frac{1}{L_{eff}} = \frac{1}{L_1} + \frac{1}{L_2} \quad ----- (8)$

Coefficient of coupling :-

When two coils are used in electric circuits they get coupled to each other and in addition to their self-inductances mutual inductive effect also comes into play. The mutual inductance between the coils is related to their self-inductance through a factor known as the 'coefficient of coupling'. It may be defined as the fraction of the magnetic flux generated by one coil that gets linked up with the other coil.

Consider two coils C_1 and C_2 of self-inductances L_1 and L_2 are coupled together. If a current I_1 through C_1 produces a flux ϕ_1, then

$$\phi_1 = L_1 I_1 .$$

If a fraction K_1 of this flux gets linked up with C_2, then the flux that is linked with coil C_2 will be

$$\phi_2 = K_1 \phi_1 = K_1 L_1 I_1 .$$

Again, if M is the mutual inductance between the coils, then we can write

$$\phi_2 = M I_1$$

$$\therefore \; M = K_1 L_1 \quad \text{----} (1)$$

Similarly if a current I_2 in C_2 produces a flux ϕ_2 and a fraction K_2 of it gets linked up with C_1 we can write

$$\phi_1 = K_2\phi_2 = K_2 L_2 I_2$$

and $\phi_1 = M I_2$

$$\therefore M = K_2 L_2 \quad \text{----} (2)$$

From eq(1) and (2) we get

$$M^2 = K_1 K_2 L_1 L_2 = K^2 L_1 L_2 \qquad [\text{where } K^2 = K_1 K_2]$$

$$\therefore M = K\sqrt{L_1 L_2} \quad \text{----} (3)$$

K is called the coefficient of coupling between the coils. The value of K lies between 0 to 1. It depends on the geometry of the two coils and their relative positions.

If $K = 1$, the coupling is said to be perfect. Here the whole flux generated by one coil gets linked up with the other coil.

If $K \lesssim 1$, the coils are said to be closely coupled.

If K is much less than 1 then the coils are said to be loosely coupled.

From eq(3) we can also define the coefficient of coupling as the ratio of mutual inductance actually present between the coils to its maximum possible value.

Energy stored in an inductor

Let an emf $\mathcal{E}$ is suddenly applied in an inductor L having resistance r. If 'i' is the current at time t after switching on, the back emf is, $-L\frac{di}{dt}$ and the net emf is, $(\mathcal{E} - L\frac{di}{dt})$

$$\therefore \mathcal{E} - L\frac{di}{dt} = ri$$

$$\text{or} \quad \mathcal{E} = ri + L\frac{di}{dt} \quad \text{-----(1)}$$

The work done by the source of emf in delivering a small charge $dq = idt$ in a further time dt is, $dW = \mathcal{E}\,idt = Li\,di + i^2 r\,dt$ [using eq(1)]

$\therefore$ Total work done in time T when the current increases from 0 to I is,

$$W = L\int_0^I i\,di + r\int_0^T i^2 dt = \frac{1}{2}LI^2 + r\int_0^T i^2 dt \, .$$

Here, the second term is the Joule heat loss in the resistance (r). The first term represents the work done against the back emf. It can be regarded as the energy stored in the inductor. This energy is completely recovered when the current is switched off.

$\therefore$ The magnetic energy stored in an inductor is $U_m = \frac{1}{2}LI^2$.