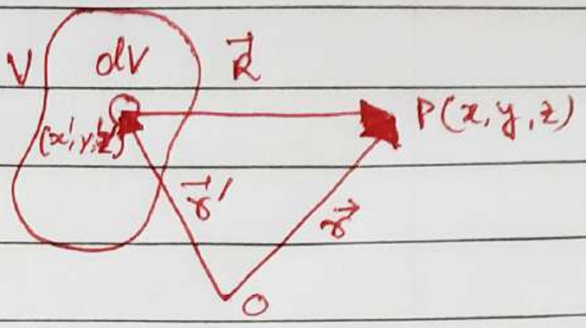
Curl of the Magnetic Field $\vec{B}$:-



According to Biot-Savart law the magnetic field $\vec{B}$ at a point $P(x,y,z)$ due to a volume distribution of current $\vec{J}(x,y,z)$ is,

$$\vec{B} = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}\times\vec{R}}{R^3}\,dV \quad \cdots\cdots (1)$$

where the integration is over the volume V, $dv = dx'dy'dz'$ and $\vec{R} = \vec{r} - \vec{r}' = \hat{i}(x-x') + \hat{j}(y-y') + \hat{k}(z-z')$.

Eq(1) ~~cat~~ can be written as,

$$\vec{B} = -\frac{\mu_0}{4\pi}\int \vec{J}\times\vec{\nabla}\left(\frac{1}{R}\right)dv \qquad \left[\because \vec{\nabla}\frac{1}{R} = -\frac{\vec{R}}{R^3}\right] \quad \cdots\cdots (2)$$

where, $\vec{\nabla}$ is with respect to the field point coordinates (x,y,z).

Now, $\vec{\nabla}\times\frac{1}{R}\vec{J} = \frac{1}{R}\vec{\nabla}\times\vec{J} + \vec{\nabla}\frac{1}{R}\times\vec{J} = \vec{\nabla}\frac{1}{R}\times\vec{J} = -\vec{J}\times\vec{\nabla}\frac{1}{R}$

[Since $\vec{J} = \vec{J}(x',y',z')$ and $\vec{\nabla}$ is with respect to the field point coordinates (x,y,z), $\vec{\nabla}\times\vec{J} = 0$]

$\therefore$ from eq(2) $$\vec{B} = \frac{\mu_0}{4\pi}\vec{\nabla}\times\int\frac{1}{R}\vec{J}\,dV \quad \cdots\cdots (3)$$

$$\therefore \vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\left[\vec{\nabla}\times\vec{\nabla}\times\int\frac{1}{R}\vec{J}\,dv\right] \qquad \left[\because \vec{\nabla}\times\vec{\nabla}\times\vec{A} = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}\right]$$

$$= \frac{\mu_0}{4\pi}\left[\vec{\nabla}\left(\vec{\nabla}\cdot\int\frac{1}{R}\vec{J}\,dv\right) - \nabla^2\int\frac{1}{R}\vec{J}\,dv\right]$$

$$= \frac{\mu_0}{4\pi}\left[\vec{\nabla}\int\vec{J}\cdot\vec{\nabla}\frac{1}{R}\,dV - \int\vec{J}\,\nabla^2\frac{1}{R}\,dV\right]$$

or, $\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\left[-\vec{\nabla}\int \vec{J}\cdot\vec{\nabla}'\frac{1}{R}\,dv + \int \vec{J}\,4\pi\delta(\vec{r}-\vec{r}')\,dv\right]$

$\left[\because \vec{\nabla}\frac{1}{R} = -\vec{\nabla}'\frac{1}{R} \text{ and } \nabla'^2\frac{1}{R} = -4\pi\delta(\vec{r}-\vec{r}')\right]$

or, $\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\left[-\vec{\nabla}\int\left\{\vec{\nabla}'\cdot\frac{1}{R}\vec{J} - \frac{1}{R}\vec{\nabla}'\cdot\vec{J}\right\}dv + 4\pi\vec{J}\right]$

or, $\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\left[-\vec{\nabla}\int\vec{\nabla}'\cdot\frac{1}{R}\vec{J}\,dv + 4\pi\vec{J}\right]$ $\quad\left[\because \vec{\nabla}'\cdot\vec{J} = 0\right]$

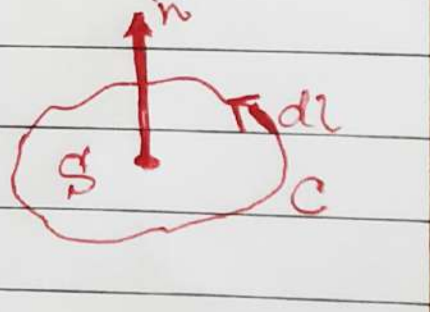
or, $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} - \frac{\mu_0}{4\pi}\vec{\nabla}\int\vec{\nabla}'\cdot\frac{1}{R}\vec{J}\,dv \quad \ldots\ldots (4)$

The integral on the right hand side can be converted to a surface integral using divergence theorem. And if the surface is chosen to lie outside the bounded region, where $\vec{J}$ is non-vanishing, then the integral vanishes.

$$\therefore \vec{\nabla}\times\vec{B} = \mu_0\vec{J}(\vec{r})$$

This is called the differential form of Ampere's law. Taking scalar surface integral through any surface S we can write,

$$\int_S (\vec{\nabla}\times\vec{B})\cdot d\vec{s} = \mu_0\int_S \vec{J}\cdot d\vec{s}$$



By applying stokes theorem

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0\int_S \vec{J}\cdot d\vec{s}.$$

where C is the closed path bounding S, $\hat{n}$ is unit vector normal to the surface S. The direction C is positive if an observer walking along C with his head pointing along positive normal to S has the surface on his left.

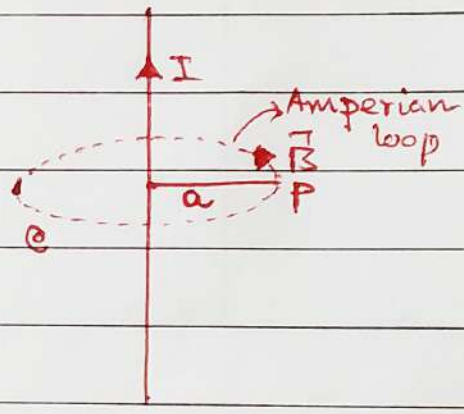
$\therefore \oint \vec{B}.d\vec{l} = \mu_0 I_{enc.}$ ...... (5)

where $I_{enc.}$ is the total current passing through the surface S enclosed by C. Eq(5) is known as the Ampere's Circuital law, also known as the integral form of the Ampere's law. In words, the line integral of the magnetic field around any closed path is equal to μ_0 times the total current enclosed by the path.

Application of Ampere's law:

(a) Long straight current carrying wire—

We consider a long straight wire carrying a current I. We want to find the magnetic field at any point P, which is at a distance a from the axis of the wire. We draw a circle



of radius a through P and with centre on the axis of the wire. It is called an Amperian loop. From the symmetry of the loop about the axis we can consider the magnitude of the magnetic field $\vec{B}$ to be same all over the path. The direction of $\vec{B}$ will be tangential to every path element $d\vec{l}$.

$$\therefore \vec{B}.d\vec{l} = B dl$$

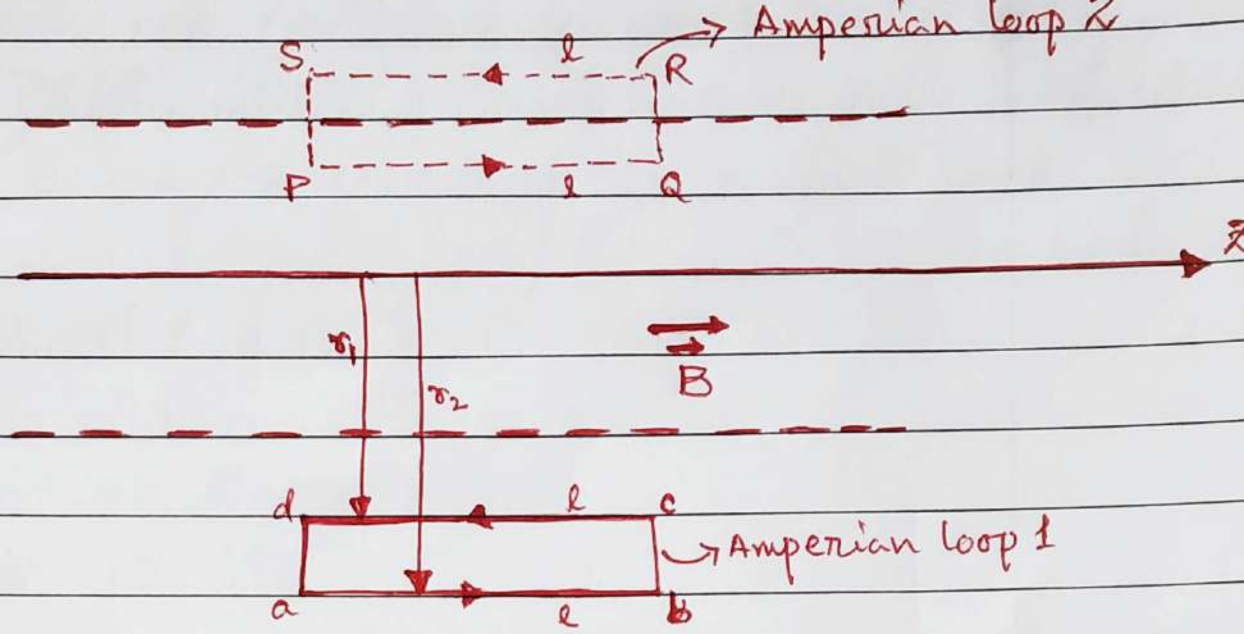
Now, from Ampere's Circuital law,

$$\oint_c \vec{B}.d\vec{l} = B\oint_c dl = 2\pi a B = \mu_0 I$$

$$\therefore B = \mu_0 I / 2\pi a$$

(b) Long Solenoid—

Let us consider a long solenoid of radius a, consisting of n turns per unit length and carrying a steady current I.



The magnetic field inside the solenoid is directed along its axis. It can't have any radial or cross radial component.

Let us consider an amperian loop in the form of a rectangle "abcd", which lies entirely outside the solenoid with its sides at distances r_1 and r_2 from the axis of the solenoid.

From Ampere's circuital law,

$$\oint \bar{B}.d\bar{l} = [B(r_2) - B(r_1)]l = \mu_0 I_{enc} = 0.$$

where $\vec{B}$ is perpendicular to ad and bc.

$$\therefore B(r_1) = B(r_2)$$

So, the magnetic field outside the solenoid does not

depend on the distance from the axis of the solenoid. Also, as $r \to \infty$, $B \to 0$.
$\therefore$ The magnetic field outside the solenoid must be zero everywhere.

Now, we consider another rectangular loop "PQRS", which is half inside and half outside the solenoid. From Ampere's circuital law,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 n l I \quad [\because I_{enc} = n l I]$$

$$\therefore \oint \vec{B} \cdot d\vec{l} = \int_{PQ} \vec{B} \cdot d\vec{l} + \int_{QR} \vec{B} \cdot d\vec{l} + \int_{RS} \vec{B} \cdot d\vec{l} + \int_{SP} \vec{B} \cdot d\vec{l}$$

$$= Bl + 0 + 0 + 0$$

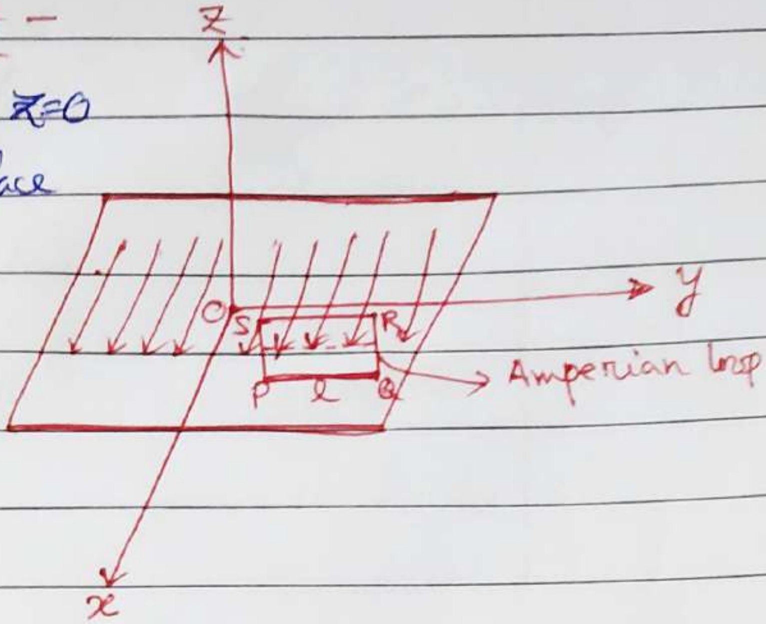
$$= Bl$$

Here B is constant over PQ, zero over RS; and zero or perpendicular over QR and SP.
$\therefore$ The magnetic field inside the solenoid is given by $B = \mu_0 n I$. It is directed along the axis of the solenoid.

$$\therefore \vec{B} = \begin{cases} \mu_0 n I \hat{z} & \text{(inside the solenoid)} \\ 0 & \text{(outside the solenoid)} \end{cases}$$

(a) Infinite sheet of Current -

Consider an infinite plane $z=0$ on which a uniform surface current density $\vec{K}=K\hat{i}$ flows. $\hat{i}$ is the unit vector along x direction. Dividing the current sheet into pairs of current filaments at



$+y$ and $-y$ it can be shown that the resultant magnetic field $\vec{B}$ will be directed along $-y$ axis for z positive and along $+y$ axis for z negative. Now, we choose a rectangle PQRS parallel to the yz plane. Applying Ampere's law we get,

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 K l$$

$$\text{or, } \int_{PQ} \bar{B}\cdot d\bar{l} + \int_{QR} \bar{B}\cdot d\bar{l} + \int_{RS} \bar{B}\cdot d\bar{l} + \int_{SP} \bar{B}\cdot d\bar{l} = \mu_0 K l$$

$$\text{or, } Bl + 0 + Bl + 0 = \mu_0 K l$$

$$\therefore B = \frac{\mu_0 K}{2}$$

Here B is independent of the distance from the current sheet.

$$\therefore \vec{B} = \begin{cases} -\frac{\mu_0 K}{2}\hat{j} & \text{for } z>0 \\ +\frac{\mu_0 K}{2}\hat{j} & \text{for } z<0 \end{cases}$$

In general $\vec{B} = \frac{\mu_0 \vec{K}\times\hat{n}}{2}$, where $\hat{n}$ is the unit vector normal from the current sheet to the point of interest.