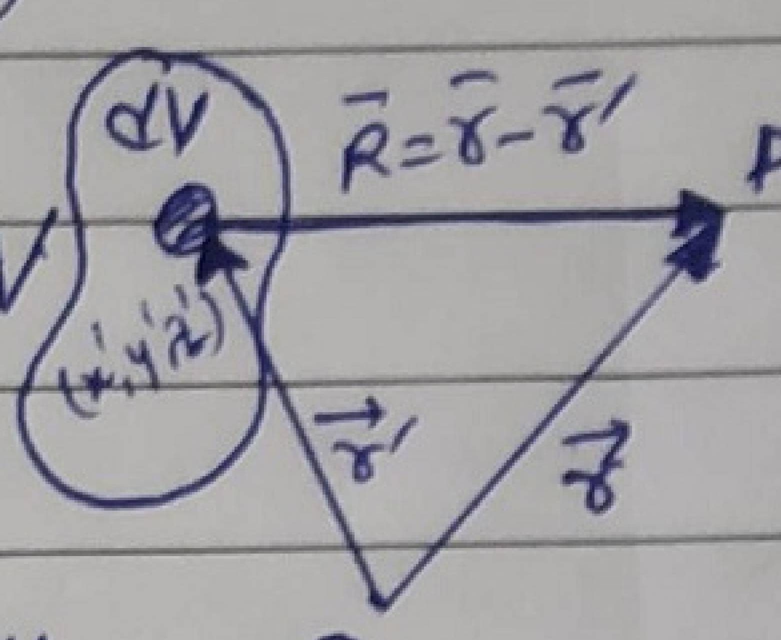
Divergence of the magnetic field $\vec{B}$:

According to Biot-Savart law the magnetic field $\vec{B}$ at any point $P(x,y,z)$ due to a volume element dv is given by $\vec{B} = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}\times\vec{R}}{R^3}\,dv$ ...... (1)

where $\vec{J}(x',y',z')$ is the current density, $\vec{R}$ is a vector from the volume element to the point of observation (P).

$\bar{R} = \bar{r} - \bar{r}' = \hat{i}(x-x') + \hat{j}(y-y') + \hat{k}(z-z')$



In eq(1) the integration is w.r.t the source point coordinates (x',y',z'). And $dv = dx'dy'dz'$.

Taking divergence of $\vec{B}$ w.r.t the field point coordinates (x,y,z) we have

$$\bar{\nabla}\cdot\vec{B} = \frac{\mu_0}{4\pi}\int_V \bar{\nabla}\cdot\frac{(\bar{J}\times\bar{R})}{R^3}\,dv \quad \cdots\cdots (2)$$

Now, $\bar{\nabla}\cdot\left(\vec{J}\times\frac{\vec{R}}{R^3}\right) = \frac{\bar{R}}{R^3}\cdot(\vec{\nabla}\times\vec{J}) - \vec{J}\cdot\left(\vec{\nabla}\times\frac{\vec{R}}{R^3}\right)$

$= 0 - \vec{J}\cdot\left[\frac{1}{R^3}(\vec{\nabla}\times\vec{R}) + \vec{\nabla}\left(\frac{1}{R^3}\right)\times\vec{R}\right]$ $[\because \vec{J}=J(x',y',z')$, $\vec{\nabla}\times\vec{J}=0]$

$= -\vec{J}\cdot\left[0 - \frac{3\vec{R}\times\vec{R}}{R^5}\right]$ $[\because \bar{\nabla}\times\bar{R}=0]$

$= -\vec{J}\cdot[0]$ $[\because \bar{R}\times\bar{R}=0]$

$= 0$

$\therefore$ from eq(2) $\bar{\nabla}\cdot\bar{B} = 0$ ..... (3)

This is true for all magnetic fields weather produced by steady or varying current.

Integrating eq(3) over any volume V and applying divergence theorem we can write

$$\int_V \vec{\nabla}\cdot\vec{B}\,dv = \oint_S \vec{B}\cdot d\vec{S} = 0$$

or, $\oint_S \vec{B}\cdot\hat{n}\,dS = 0$ .... (4)

where S is the surface bounding the volume V. Equations (3) and (4) are known as Gauss's law for magnetism. The quantity

$$\Phi = \int_{S_1} \vec{B}\cdot\hat{n}\,dS$$

is called the magnetic flux through the surface S_1, and in SI units measured in weber (wb).

Eq (4) shows that the flux of $\vec{B}$ through any closed surface 'S' is always zero. That is magnetic lines of force are always closed and there is no isolated magnetic monopole or free magnetic charge. Magnetic poles are equal and opposite and occur in pairs. The magnetic lines of force are continuous and there is no magnetic monopole in which these lines originates or terminates.

A field for which $\vec{\nabla}\cdot\vec{B} = 0$ is called a solenoidal field. Because the field lines close upon themselves as in case of the field of a solenoid and there is no net flux through any closed surface.

Magnetic Vector Potential:

We Know that for any magnetic field $\vec{B}$, $\vec{\nabla}\cdot\vec{B}=0$. Since the divergence of the curl of a vector is always zero, we can write

$$\vec{B}=\vec{\nabla}\times\vec{A} \quad \text{-----} \quad (1)$$

where, $\vec{A}$ is called the magnetic vector potential. The SI unit of magnetic vector potential is weber per meter (wb/m).

$$[\bar{\nabla}\cdot\vec{B}=\bar{\nabla}\cdot(\bar{\nabla}\times\bar{A})=0]$$

Also, starting from Biot-Savart law one can obtain the curl of the magnetic field $\vec{B}$ as,

$$\bar{\nabla}\times\vec{B}=\mu_0\vec{J} \quad \text{----} \quad (2)$$

where μ_0 is permeability of free space and $\vec{J}(\bar{r})$ is the current density. Eq(2) is the differential form of Ampere's law.

$$\therefore \bar{\nabla}\times\vec{B}=\bar{\nabla}\times(\bar{\nabla}\times\bar{A})=\mu_0\bar{J}$$

$$\text{or, } \vec{\nabla}(\bar{\nabla}\cdot\vec{A})-\bar{\nabla}^2\vec{A}=\mu_0\bar{J} \quad \text{----} \quad (3)$$

The vector potential $\vec{A}$ defined by eq(1) is not unique. Because we can add to $\vec{A}$ any function whose curl vanishes, with no effect on $\vec{B}$. This may be used to eliminate the divergence of $\vec{A}$.

$$\therefore \vec{\nabla}\cdot\vec{A}=0$$

$\therefore$ From eq(3) $\nabla^2\vec{A}=-\mu_0\vec{J}$ ---- (4)

This is vector Poisson's equation.

If the cartesian component of $\vec{A}$ and $\vec{J}$ are respectively A_x, A_y, A_z and J_x, J_y, J_z then equ(4) gives $\nabla^2 A_x = -\mu_0 J_x$, $\nabla^2 A_y = -\mu_0 J_y$, $\nabla^2 A_z = -\mu_0 J_z$.
These equations has the form of Poisson's equation $\nabla^2\phi = -\rho/\epsilon_0$ in electrostatics, which has the solution

$$\phi(\bar{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\bar{r}_0)\,dV_0}{|\bar{r}-\bar{r}_0|} \quad \text{----} \quad (5)$$

where, ρ is the charge density, ϵ_0 is the permittivity of free space and ϕ is the electrostatic potential.
Hence the solutions for the cartesian components of $\bar{A}$ may be written as,

$$A_i(\bar{r}) = \frac{\mu_0}{4\pi}\int \frac{J_i(\bar{r}_0)\,dV_0}{|\bar{r}-\bar{r}_0|},$$

i stands for x, y, z and '0' stands for source point coordinates.

$$\therefore \bar{A}(\bar{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\bar{r}_0)\,dV_0}{|\bar{r}-\bar{r}_0|} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\,dV}{R}, \qquad R = |\bar{r}-\bar{r}_0|$$

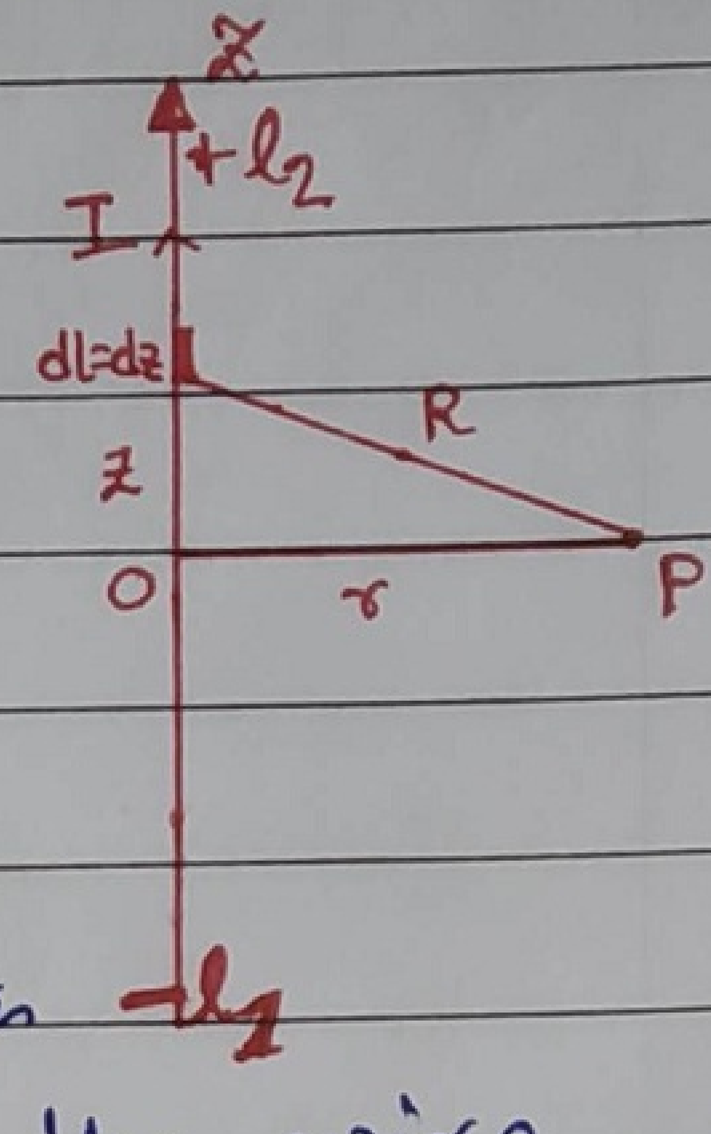
For line current, $\vec{A} = \frac{\mu_0}{4\pi}\int \frac{I\,d\vec{l}}{R}$

For surface current, $\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}\,ds}{R}$

$\vec{K}$ is the surface current density.

Calculation of $\vec{A}$ in some simple cases:

1. We consider a long straight wire carrying a steady current I. We will calculate the magnetic vector potential $\bar{A}$ at a point P which is at a distance r from the wire.



We choose the cylindrical coordinates (r, θ, z) with z-axis along the wire. O is the foot of the perpendicular from P on the wire. O is the origin of the coordinate system.

Now we consider an element $d\vec{l} = \hat{z}\,dz$ at a distance z from O.

$$\therefore \vec{A} = \frac{\mu_0}{4\pi}\int \frac{I\,d\vec{l}}{R} = \hat{z}\frac{\mu_0 I}{4\pi}\int_{-l_1}^{l_2} \frac{dz}{\sqrt{r^2+z^2}}$$

$$\text{or, } \vec{A} = \hat{z}\frac{\mu_0 I}{4\pi} \ln\left(z+\sqrt{r^2+z^2}\right)\Big|_{-l_1}^{l_2}$$

$$\therefore \vec{A} = \hat{z}\frac{\mu_0 I}{4\pi} \ln\left[\frac{l_2+\sqrt{r^2+l_2^2}}{-l_1+\sqrt{r^2+l_1^2}}\right] \quad \text{------ (1)}$$

If the wire is of length 2L and P is just above the centre of the wire, then $l_1 = l_2 = L$ and we have,

$$\vec{A} = \hat{z}\frac{\mu_0 I}{4\pi} \ln\left[\frac{L+\sqrt{r^2+L^2}}{-L+\sqrt{r^2+L^2}}\right] \quad \text{---- (2)}$$

or, $$\vec{A} = \hat{z}\frac{\mu_0 I}{4\pi}\ln\left[\frac{1+\left[1+\left(\frac{r}{L}\right)^2\right]^{1/2}}{-1+\left[1+\left(\frac{r}{L}\right)^2\right]^{1/2}}\right]$$

If $r \ll L$, $\frac{r}{L} \ll 1$

$$\therefore \vec{A} \simeq \hat{z}\frac{\mu_0 I}{4\pi}\ln\left[\frac{1+1+\frac{1}{2}\left(\frac{r}{L}\right)^2}{\cancel{-1}+\cancel{1}+\frac{1}{2}\left(\frac{r}{L}\right)^2}\right]$$

$$= \hat{z}\frac{\mu_0 I}{4\pi}\ln\left[\frac{2+\frac{r^2}{2L^2}}{\frac{r^2}{2L^2}}\right]$$

$$= \hat{z}\frac{\mu_0 I}{4\pi}\ln\left(1+\frac{4L^2}{r^2}\right)$$

Now, $\frac{L^2}{r^2} \gg 1$

$$\cancel{\therefore \vec{A} \simeq \hat{z}\frac{\mu_0 I}{4\pi}\ln\left[\left(\frac{2L^2}{r^2}\right)\right]}$$

$$\therefore \vec{A} \simeq \hat{z}\frac{\mu_0 I}{4\pi}\ln\left[\frac{4L^2}{r^2}\right] = \hat{z}\frac{\mu_0 I}{2\pi}\ln\left(\frac{2L}{r}\right) \quad \ldots\ldots (3)$$

Also, the magnetic field, $\vec{B} = \vec{\nabla}\times\vec{A}$

or, $$\vec{B} = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & 0 & A_z \end{vmatrix} = -\hat{\theta}\frac{\partial A_z}{\partial r}$$

or, $$\vec{B} = -\hat{\theta}\frac{\partial}{\partial r}\left[\frac{\mu_0 I}{2\pi}\ln\left(\frac{2L}{r}\right)\right] = \hat{\theta}\frac{\mu_0 I}{2\pi r} \quad \ldots (4)$$

2 Long Solenoid:

We consider an infinite solenoid with n turns per unit length, radius a and carrying current I. Symmetry of the problem suggests that $\vec{A}$ should have the direction of current, i.e. the circumferential direction $\hat{\theta}$. The magnetic field inside the solenoid is along the axis (z-axis) of the solenoid and we have

$$\vec{B} = \mu_0 n I \hat{z}$$

If we consider a circular Amperian loop of radius r ($r<a$) inside the solenoid in xy plane we have

$$\oint_C \vec{A}\cdot d\vec{l} = \int_S (\vec{\nabla}\times\vec{A})\cdot d\vec{S} = \int_S \vec{B}\cdot d\vec{S}$$

Applying Stoke's theorem

$$\text{or, } A.2\pi r = \mu_0 n I . \pi r^2$$

$$\therefore A = \frac{\mu_0 n I}{2} r \quad \text{for } r<a \quad \cdots\cdots (1)$$

Now, for a circular Amperian loop of radius r ($r>a$) outside the solenoid,

$$\oint_C \vec{A}\cdot d\vec{l} = \int_S \vec{B}\cdot d\vec{S} = \mu_0 n I \pi a^2$$

[Since the field only extends upto $r=a$. Outside, the field is zero.]

$$\text{or, } A.2\pi r = \mu_0 n I \pi a^2$$

$$\therefore A = \frac{\mu_0 n I a^2}{2r} \quad \text{for } r>a \quad \cdots\cdots (2)$$