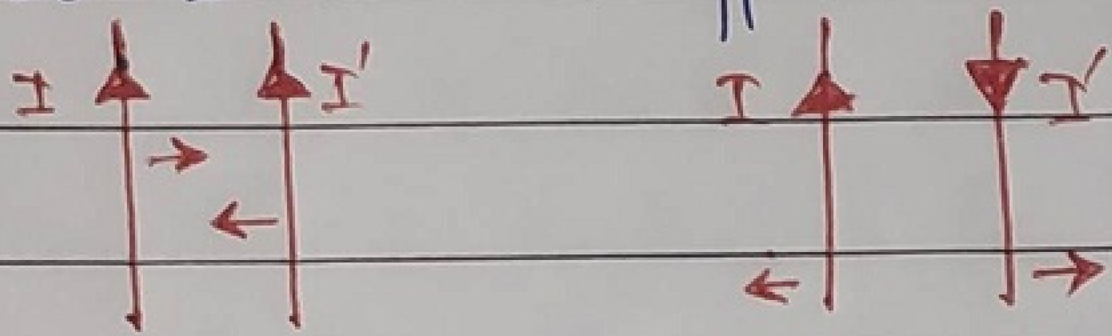
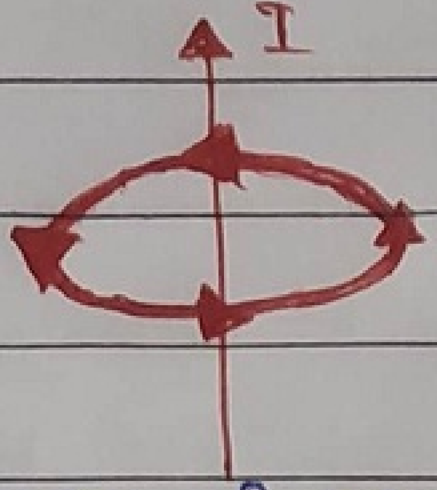
Electricity and Magnetism

Unit 6: The Magnetostatic Field

In electrostatics we study the behaviour of electric charges at rest. However, things are different if we consider the charges are in motion. If we run currents through two wires in parallel, we find that they are attracted when the currents run in the same direction, and they are repulsed when the currents run in opposite directions.



This force is proportional to the currents. If we double the current in one wire the force will be double. If we double the currents in both wires the force will be quadruple. There indicate that the force is proportional to the velocity of the moving charge and it points in a direction perpendicular to the velocity. It also indicates that the force depends on a "cross product". Here what we say is that a magnetic field $\vec{B}$ arises from the current (we will talk about this in more detail later). The direction of this field in kind of odd. It wraps around the current in a circular fashion, with a direction defined by the right-hand rule. We point our right thumb in the direction of the current and our fingers curl in the same sense as the magnetic field.



With this sense of magnetic field defined, the force that arises when a charge 'q' moves with velocity $\vec{v}$ through this field is given by,

$$\vec{F}_m = q(\vec{v} \times \vec{B}) \text{ --- (1) (In SI units)}$$

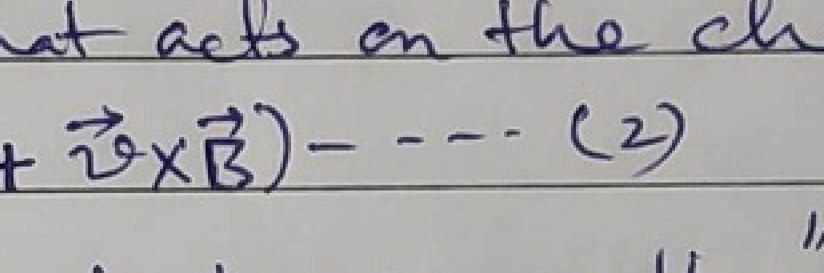
In addition if an electric field $\vec{E}$ is also present then, the total force that acts on the charge is

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \text{ ---- (2)}$$

This combined force is known as the "Lorentz force."

The SI unit of the magnetic field is Tesla (T).

If I shoot a charge into a region of uniform magnetic field, such that the field points out of the paper and the velocity is initially points to the right, what motion results from the magnetic force?



At every instant the magnetic force points perpendicular to the charge's velocity - exactly the force needed to cause circular motion. If the particle has charge q and mass m, then

$F_m = F_{centripetal}$ or, $qvB = mv^2/R$

$\therefore R = mv/qB$ ---- (3). [R is the radius of the circle]

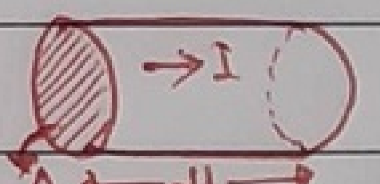
If the charge q is positive, the particle's trajectory suddenly changes direction to the right. On the other hand if q is negative then the particle will change direction to the left.

From eq(1) it is clear that the magnetic field does not exert any force on an electric charge at rest (i.e $v=0$). Magnetic force $\vec{F}_m$ is perpendicular to the velocity $\vec{v}$ of the charge. So, it cannot do any work on the charge. Work done by $\vec{F}_m$ on q as it moves a distance $\vec{v}dt$ is

$$\vec{F}_m \cdot \vec{v}dt = 0$$

Magnetic force on a current carrying conductor.

Suppose a conducting wire is placed in a magnetic field $\vec{B}$. Consider an element $d\vec{l}$ of the conductor carrying a current I.



The direction of $d\vec{l}$ is that of the current I. Therefore $d\vec{l}$ is parallel to the velocity $\vec{v}$ of the charge carriers.

Here, the number of charge carriers inside the element dl is NAdl, where A is the cross-sectional area of the conductor and N is the number of charge carriers per unit volume.

Each of the carriers experiences a magnetic force $q(\vec{v}\times\vec{B})$.

$\therefore$ Total force on all the charge carriers is

$$d\vec{F} = NAdl\,q(\bar{v}\times\bar{B}) = NAvq\,(d\vec{l}\times\vec{B})$$

$[\because d\bar{l}$ is parallel to $\bar{v}]$

or, $d\vec{F} = I(d\vec{l}\times\vec{B})$

Here $I = NqAv$ is the current through the wire.

For a conductor of finite length total force is

$$\vec{F} = I\int(d\vec{l}\times\vec{B}). \quad ----(4)$$

For a wire of length l placed in a uniform magnetic field $\bar{B}$, the force is $\bar{F} = I(\vec{l}\times\vec{B})$.

For a closed loop placed in a uniform magnetic field $\vec{B}$, $\vec{F} = I\oint d\vec{l}\times\vec{B} = 0$. $\quad [\because \oint d\bar{l} = 0]$

Biot-Savart Law:

According to Biot-Savart law the magnetic field at any point due to an element $d\vec{l}$ of a conductor carrying current I is

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\bar{l}\times\bar{R}}{R^3} \quad \ldots\ldots (5)$$

Here, $\bar{R}$ is the vector from the element $d\bar{l}$ to the point of observation. μ_0 is the permeability of free space ($\mu_0 = 4\pi\times10^{-7}\ N/A^2$).

If $\vec{J}$ is the current density. we can write

$$I\,d\vec{l} = (\vec{J}.\hat{n}\,dS)\,d\bar{l} = (d\vec{l}.\hat{n}\,dS)\,\vec{J} = \vec{J}\,dV$$

$[\because \bar{J}$ is parallel to $d\bar{l}]$

where dS is the cross-sectional area of the conductor, $\hat{n}$ is a unit vector normal to dS, and dV is the volume of the conductor of length dl.

Now, from eq(5) we have,

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{\vec{J} \times \vec{R}}{R^3} dV \quad \text{----- (6)}$$

This is another form of Biot-Savart law.

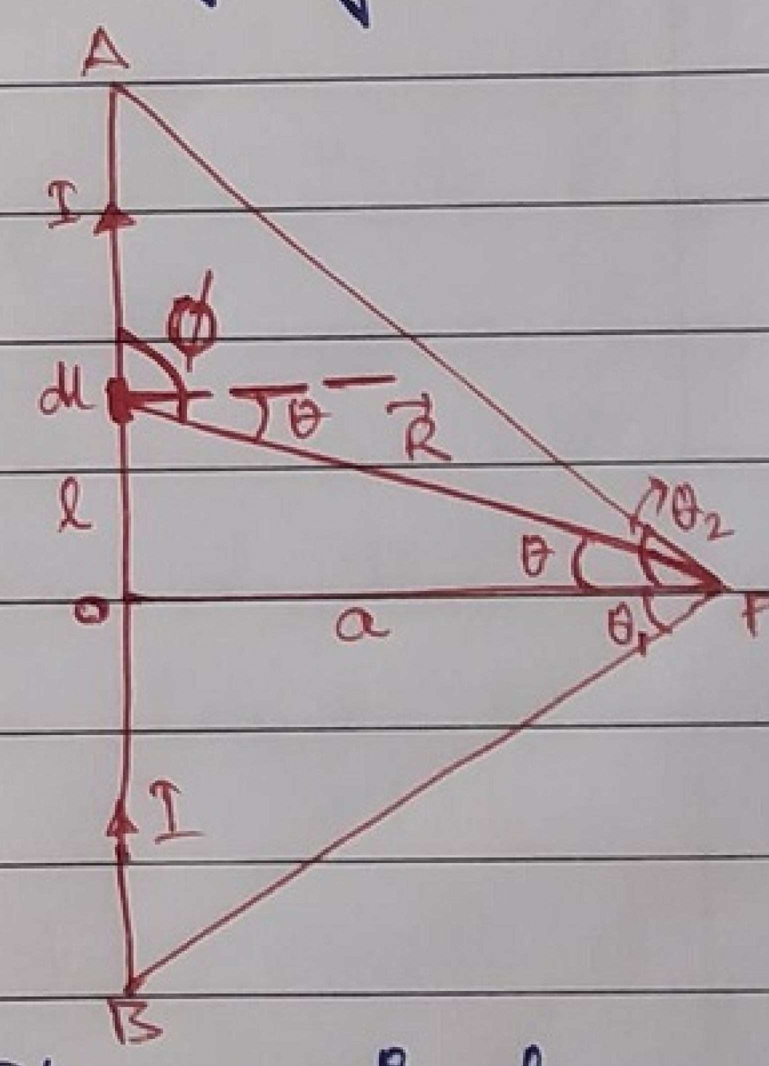
For surface current Biot-Savart law can be written as, $d\vec{B} = \frac{\mu_0}{4\pi} \frac{\vec{K} \times \vec{R}}{R^3} dS$ ----- (7)

Where $\vec{K}$ is the surface current density over an elemental surface area dS.

Application of Biot-Savart law

1. Long straight current carrying conductor:

Consider a long straight wire carrying a current I. We will find the magnetic field $|\vec{B}|$ at a point P, which is at a distance 'a' from the wire. OP is the normal from P and on AB.



Let us now consider a small element $d\vec{l}$ at a distance l from O. According to Biot-Savart law the magnetic field at P due to the element $d\vec{l}$ is,

$$d\vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{I d\vec{l} \times \vec{R}}{R^3}$$

Here $\vec{B}$ is normal to the plane containing $d\vec{l}$ and $\vec{R}$. It is actually inward to the plane of the paper.

∴ The total field due to the entire wire is,

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int \frac{d\bar{l}\times\bar{R}}{R^3}.$$

Here, $|d\vec{l}\times\vec{R}| = dl.R.\sin\phi = dl.R\sin(90^\circ+\theta) = dl.R\cos\theta$

Also, $l = a\tan\theta$. $\therefore dl = a\sec^2\theta d\theta = a\left(\frac{R}{a}\right)^2 d\theta = \frac{R^2}{a}d\theta$

So, $|d\bar{l}\times\bar{R}| = \frac{R^2}{a}d\theta \times R\cos\theta = \frac{R^3}{a}\cos\theta d\theta$

or, $\frac{|d\bar{l}\times\bar{R}|}{R^3} = \frac{1}{a}\cos\theta d\theta$

$$\therefore |\bar{B}| = \frac{\mu_0 I}{4\pi}\int\frac{|d\bar{l}\times\bar{R}|}{R^3} = \frac{\mu_0 I}{4\pi}\int_{-\theta_1}^{\theta_2}\frac{1}{a}\cos\theta\, d\theta$$

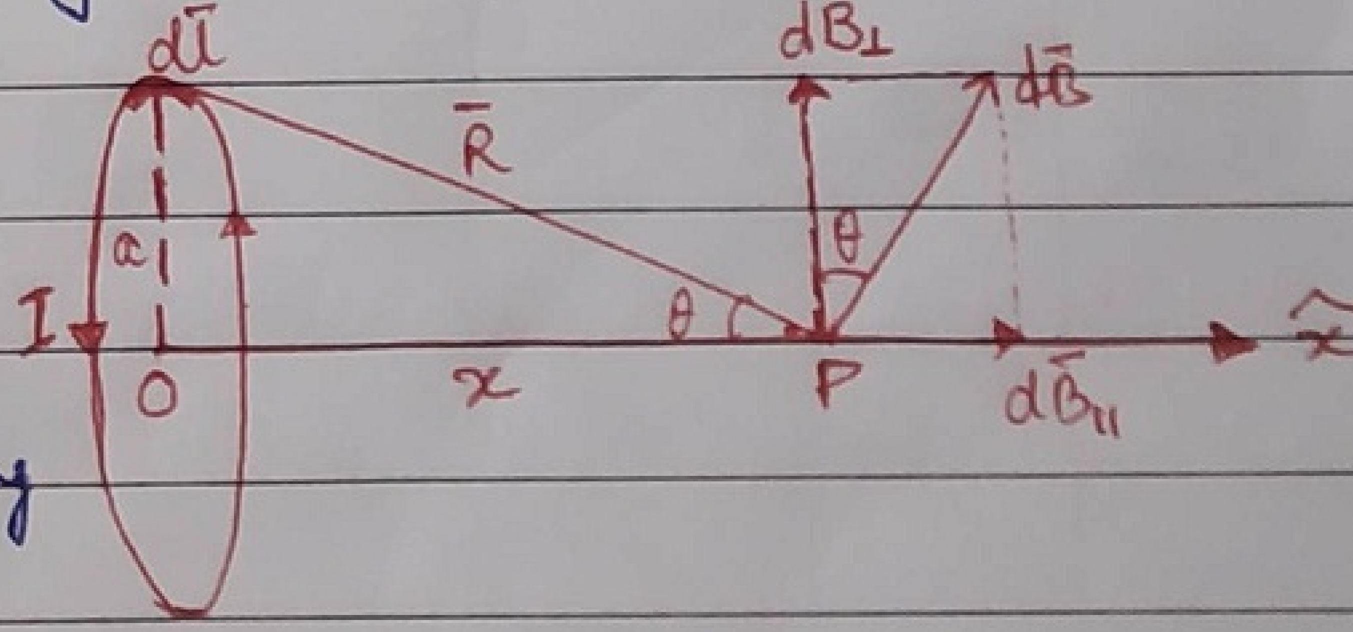
or, $|\vec{B}| = \frac{\mu_0 I}{4\pi a}(\sin\theta_2 + \sin\theta_1)$.

For an infinite long wire $\theta_1 = \theta_2 = \pi/2$,

and $|\bar{B}| = \mu_0 I/2\pi a$

The direction of the magnetic field $\bar{B}$ at any point P, which is at a distance a from the wire, is tangential to the circle of radius a. The centre of the circle is on the wire. The circle will be lying on a plane perpendicular to the wire.

2. Circular Coil:



We consider a circular loop of radius a, carrying a current I. We will calculate the magnetic field $|\bar{B}|$ at any axial point P at a distance x from the centre (O) of the loop.

The magnetic field at P due to an element $d\vec{l}$ of the loop is, $d\vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{I d\vec{l} \times \vec{R}}{R^3}$.

The field is perpendicular to the plane containing $d\bar{l}$ and $\bar{R}$. For the entire coil $d\bar{B}$ will describe a cone with OP as axis. As a result, for the entire coil the components of the field normal to OP cancel out, while the components parallel to OP combine to give a resultant magnetic field along OP.

The component of $d\bar{B}$ along OP (i.e x-axis) is, $d\bar{B}_{\parallel} = \frac{\mu_0 I}{4\pi} \frac{d\bar{l} \times \bar{R}}{R^3} \sin\theta$.

For the entire coil, we can write the total field at P as,

$$|\bar{B}| = \frac{\mu_0 I}{4\pi} \int \frac{|d\bar{l} \times \bar{R}|}{R^3} \sin\theta$$

$$= \frac{\mu_0 I}{4\pi} \int \frac{dl . R . \sin 90^\circ}{R^3} \sin\theta \qquad [\because d\bar{l} \text{ is perpendicular to } \bar{R}]$$

$$= \frac{\mu_0 I}{4\pi} \int \frac{dl . a}{R^3} \qquad [\because \sin\theta = \frac{a}{R}]$$

Now, since R is constant here, we have

$$|\bar{B}| = \frac{\mu_0 I a}{4\pi R^3} \int dl = \frac{\mu_0 I a}{4\pi R^3} . 2\pi a .$$

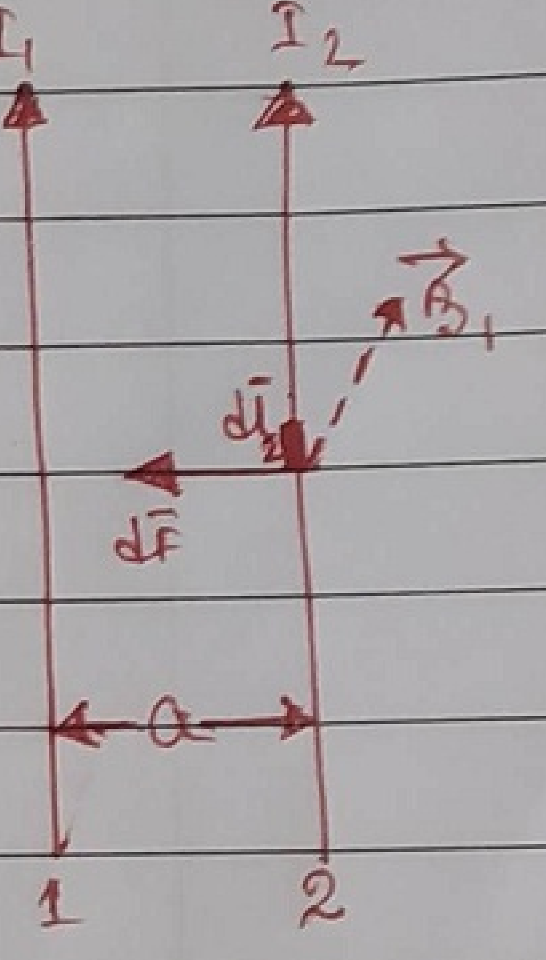
$$\therefore |\bar{B}| = \frac{\mu_0 I a^2}{2R^3} = \frac{\mu_0 I a^2}{2(a^2+x^2)^{3/2}}$$

$$[\text{since } R^2 = a^2 + x^2]$$

For a coil of N turns the field will be $\frac{\mu_0 I a^2 N}{2(a^2+x^2)^{3/2}}$

The magnetic field at the centre of the coil $(x=0)$ of N turns will be $\mu_0 I N/2a$.

Force between two long straight parallel current carrying conductors:



Let us consider two long straight parallel wires carrying currents I_1 and I_2, and are separated by a distance a. The conductor 1 produces a magnetic field at the position of conductor 2,

$$B_1 = \mu_0 I_1 / 2\pi a \quad \text{-----} \quad (1)$$

The direction of B_1 is at right angles to the plane of the wires. The force $d\bar{F}$ acting on the element $d\bar{l}_2$ of the conductor 2 due to $\vec{B}_1$ is,

$$d\bar{F} = I_2 (d\bar{l}_2 \times \bar{B}_1)$$

$$\therefore |d\bar{F}| = \frac{\mu_0}{2\pi} \cdot \frac{I_1 I_2}{a} \cdot dl_2$$

It is directed towards the conductor 1, if I_1 and I_2 flow in the same direction. If the currents are in opposite direction then the force is directed away from the conductor 1.

Here the magnitude of the force per unit length of the conductor is,

$$|\bar{F}| = \frac{\mu_0}{2\pi} \cdot \frac{I_1 I_2}{a}$$