FOURIER TRANSFORMS

Fourier series representation is only valid for a periodic function in a fixed interval. If we wish to obtain such representation for functions defined over an infinite interval with no particular periodicity, we use "integral transform". "Fourier transform" is one of the integral transform.

The Fourier transform provides a representation of functions defined over an infinite interval and having no particular peridicity, in terms of superposition of sinusoidal functions. It may be considered as generalisation of the Fourier series representation.

Since Fourier transforms are often used to represent time-varying functions, we shall denote it as a function of time, such as $f(t)$.

Our only requirement on $f(t)$ is that $\int_{-\infty}^{+\infty} |f(t)|\,dt$ is finite.

- **Transition from Fourier series to Fourier transform:**

Recall that, a function of period T may be represented as a complex Fourier series as:

$$f(t) = \sum_{r=-\infty}^{\infty} C_r\, e^{2\pi i r t/T}$$

If $T \to \infty$ (i.e. no periodicity), the infinite sum in the Fourier series will become an integral and the coefficients C_r will become functions of continous variable $\omega = 2\pi/T$.

Then we can define the Fourier transform of $f(t)$ by

$$\mathcal{F}[f(t)] = \boxed{\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t}\, dt} \leftarrow \text{F.T.}$$

and its inverse by

$$\boxed{f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{f}(\omega)\, e^{i\omega t}\, d\omega} \leftarrow \text{I.F.T.}$$

(*) Note that, introducing the constant $\frac{1}{\sqrt{2\pi}}$ in the definitions of $\tilde{f}(\omega)$ or $f(t)$ is completely arbitrary, the only requirement is that the product of these constant in $\tilde{f}(\omega)$ and $f(t)$ should be equal to $\frac{1}{2\pi}$.

Ex 1: Find the Fourier transform of the exponential decay function defined as:

$$f(t) = \begin{cases} 0 & \text{for } t<0 \\ Ae^{-\lambda t} & \text{for } t \geq 0 \end{cases}$$

where $\lambda > 0$.

Sol^n:

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} (0) e^{-i\omega t} dt + \frac{A}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\lambda t} e^{-i\omega t} dt$$

$$= 0 + \frac{A}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(\lambda + i\omega)t} dt$$

$$= \frac{A}{\sqrt{2\pi}} \left[\frac{e^{-(\lambda+i\omega)t}}{-(\lambda+i\omega)} \right]_0^{\infty}$$

$$= \frac{A}{\sqrt{2\pi}\,(\lambda + i\omega)}.$$

[Note that, this $\tilde{f}(\omega)$ can be verified by using the inverse Fourier transform to get $f(t)$ — but evaluation of this integral requires the use of contour integration.]

Ex 2: Find the Fourier transform of the normalised Gaussian distribution:

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{t^2}{2\sigma^2}\right), \quad -\infty < t < \infty$$

Sol^n: This Gaussian distribution is centred at $t=0$ and has a "root mean square deviation" σ (or σ^2 is called the variance).

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{t^2}{2\sigma^2}\right) \exp(-i\omega t)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}\left[t^2 + 2\sigma^2 i\omega t + (\sigma^2 i\omega)^2 - (\sigma^2 i\omega)^2\right]\right\} dt$$

[where the quantity $-\frac{(\sigma^2 i\omega)^2}{2\sigma^2}$ has been both added and subtracted in order to allow the factors in the integral to be expressed as a square]

$$= \frac{1}{\sqrt{2\pi}} \exp\left[\frac{(\sigma^2 i\omega)^2}{2\sigma^2}\right] \cdot \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left[-\frac{(t+\sigma^2 i\omega)^2}{2\sigma^2}\right] \cdot dt$$

Now change the variable $z = t + \sigma^2 i\omega$

so, $dz = dt$.

$$= \frac{1}{\sqrt{2\pi}} \exp\left[\frac{(\sigma^2 i\omega)^2}{2\sigma^2}\right] \cdot \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left[-\frac{z^2}{2\sigma^2}\right] dz.$$

Normalization of Gaussian integral gives

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{z^2}{2\sigma^2}\right) dz = 1.$$

So, $$\boxed{\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\sigma^2\omega^2}{2}\right)}$$

which is another Gaussian distribution, centred at $\omega = 0$ and with a root mean square deviation $= \frac{1}{\sigma}$.

Note that, the root mean square deviation in t was σ and now that in ω is $\frac{1}{\sigma}$. So it is seen that the deviations or "spreads" in t and in ω are inversely related!

$\Delta\omega\, \Delta t = 1.$ ← independently of the value of σ.

σ is also called the standard deviation and is defined as;

$$\sigma = \sqrt{\overline{(x-\mu)^2}} \quad \text{(where } \overline{} \text{ denotes average or expectation value.)}$$

$$= \sqrt{\overline{(x^2)} - (\overline{x})^2} \quad (\text{where } \mu = \overline{x})$$

→ This is known as "uncertainty principle".

Some useful formulas used in computation of Fourier transform:

1. $$\sin at = \frac{e^{iat} - e^{-iat}}{2i}$$

$$\cos at = \frac{e^{iat} + e^{-iat}}{2}$$

$$e^{iat} = \cos(at) + i\sin(at)$$
$$e^{-iat} = \cos(at) - i\sin(at)$$

2. $$\int_0^\infty e^{-at} \sin \lambda t \, dt = \frac{\lambda}{a^2+\lambda^2}$$

$$\int_0^\infty e^{-at} \cos \lambda t \, dt = \frac{a}{a^2+\lambda^2}$$

3. $$\int_0^\infty \frac{\sin \lambda t}{t} \, dt = \frac{\pi}{2}, \text{ when } \lambda > 0$$

4. $$\int_0^\infty e^{-a^2t^2} \, dt = \frac{\sqrt{\pi}}{2a}$$

Dirichlet's conditions for existence of Fourier transform:

Fourier transform can be applied to any function $f(t)$ if it satisfies the following conditions:

(i) $f(t)$ is absolutely integrable, i.e. $\int_{-\infty}^{+\infty} |f(t)| \, dt$ is convergent.

(ii) $f(t)$ has a finite number of maxima and minima.

(iii) $f(t)$ has only a finite number of discontinuities.

Ex3: Find the Fourier transform of

$$F(t) = \begin{cases} 1-t^2, & |t|<1 \\ 0, & |t|>1 \end{cases}$$

Solⁿ:

$$\tilde{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(t)\, e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{+1} (1-t^2)\, e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \left[(1-t^2)\frac{e^{-i\omega t}}{(-i\omega)} \Bigg|_{-1}^{+1} - (-2)\int_{-1}^{+1} t\, \frac{e^{-i\omega t}}{(-i\omega)}\, dt \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{-2}{i\omega} \int_{-1}^{+1} t\, e^{-i\omega t}\, dt \right]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{(-2)}{i\omega} \left[t\, \frac{e^{-i\omega t}}{(-i\omega)} \Bigg|_{-1}^{+1} - \int_{-1}^{+1} \frac{e^{-i\omega t}}{(-i\omega)}\, dt \right]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{(-2)}{i\omega} \left[\frac{e^{-i\omega}}{-i\omega} + \frac{e^{i\omega}}{-i\omega} - \frac{1}{(-i\omega)^2} e^{-i\omega t} \Bigg|_{-1}^{+1} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{(-2)}{i\omega} \left[\frac{1}{i\omega}\left(-e^{-i\omega} - e^{i\omega}\right) + \frac{1}{\omega^2}\left(e^{-i\omega} - e^{i\omega}\right) \right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left[-\frac{1}{\omega^2}\left(e^{-i\omega} + e^{i\omega}\right) - \frac{1}{i\omega^3}\left(e^{-i\omega} - e^{i\omega}\right) \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{2}{\omega^2}\cos\omega + \frac{2}{\omega^3}\sin\omega \right]$$

$$= \frac{2\sqrt{2}}{\sqrt{\pi}} \left(\frac{\sin\omega - \omega\cos\omega}{\omega^3} \right).$$

The Dirac δ-function and its relation to Fourier transform:

- **Definitions:** The Dirac ~~def~~ δ-~~function~~ can be "loosely" thought as a function which is zero everywhere except at the origin, where it is infinite.

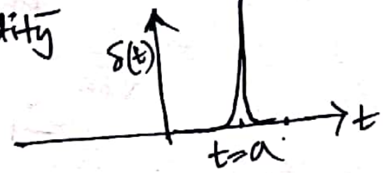
$$\delta(t) = \begin{cases} +\infty & \text{if } t=0 \\ 0 & \text{, if } t\neq 0 \end{cases}$$

or,

$$\delta(t-a) = \begin{cases} +\infty & \text{if } t=a \\ 0 & \text{if } t\neq a. \end{cases}$$

and it is also constrained to satisfy the identity

$$\int_{-\infty}^{+\infty} \delta(t)\, dt = 1.$$



—— Remember that, this is merely a heuristic characterization. It's not a function in traditional sense, rather it is a "distribution" or a "measure".

- Its fundamental defining property is

$$\boxed{\int f(t)\, \delta(t-a)\, dt = f(a)}$$

⊛ provided the range of integration includes the point $t=a$, ⊛ otherwise, the integral equals to zero.

This follows that: $\int \delta(t-a)\, dt = 1$, provided the range of integration includes $t=a$.

- **Useful properties:**

$$\delta(t) = \delta(-t)$$

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

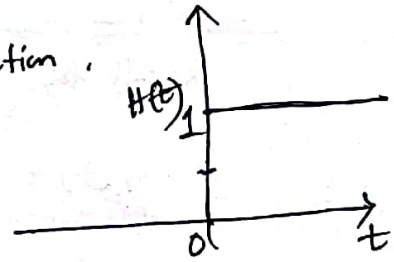
$$t\,\delta(t) = 0$$

• Heaviside step function : (Unit step function)

Closely related to the Dirac δ-function.

Defined as:

$$H(t) = \begin{cases} 1 & \text{for } t > 0 \\ 1/2 & \text{for } t = 0 \\ 0 & \text{for } t < 0. \end{cases}$$



This function is clearly discontinuous at $t=0$, and by convention, we can take $H(0) = \frac{1}{2}$.

Prove that: $\boxed{H'(t) = \delta(t)}$ where $' \equiv \frac{d}{dt}$.

Ans:

$$\int_{-\infty}^{+\infty} f(t)\, H'(t)\, dt = \Big[f(t)\, H(t) \Big]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f'(t)\, H(t)\, dt$$

$$= f(\infty) - \int_{0}^{\infty} f'(t)\, dt$$

$$= f(\infty) - \Big[f(t) \Big]_{0}^{\infty}$$

$$= f(0)$$

Now compare this with the relation we wrote earlier i.e.

$$\int_{-\infty}^{+\infty} f(t)\, \delta(t)\, dt = f(0) \quad (\text{in this case } a=0)$$

↳ fundamental property of Dirac δ-function.

Thus, $$H'(t) = \delta(t).$$

- **Relation of the δ-function to Fourier transform:**

Recall the Fourier inversion formula

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{f}(\omega) e^{i\omega t} \, d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega \, e^{i\omega t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} du \, f(u) e^{-i\omega u} \quad \left(\text{using F.T. formula of } \tilde{f}(\omega)\right)$$

$$= \int_{-\infty}^{+\infty} du \, f(u) \left\{ \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega(t-u)} d\omega \right\}$$

Now compare this with the fundamental defining property of Dirac δ-function,

$$\int f(t) \, \delta(t-a) \, dt = f(a)$$

in this case, $\int f(u) \, \delta(u-t) \, du = f(t)$.

So, we can write,

$$\boxed{\delta(u-t) = \delta(t-u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega(t-u)} d\omega}$$

- **Another representation of δ-function:**

$$\boxed{\delta(t) = \lim_{k \to \infty} \left(\frac{\sin kt}{\pi t} \right)}$$

- Special case, for $u=0$, $\boxed{\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega t} \, d\omega}$

So, $$\delta^*(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega t} \, d\omega = \delta(-t) = \delta(t)$$

Thus, δ-function is real.

- **Fourier transform of a δ-function:**

$$\boxed{\tilde{\delta}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \delta(t) e^{-i\omega t} \, dt = \frac{1}{\sqrt{2\pi}}}$$

Properties of Fourier transforms:

- Notation: We denote the Fourier transform of $F(t)$ by $\tilde{f}(\omega)$ or $\mathcal{F}[F(t)]$

All these properties can be derived from the definition of the Fourier transform.

1) Differentiation:

$$\mathcal{F}[f'(t)] = i\omega \tilde{f}(\omega) \quad \text{or} \quad i\omega\, \mathcal{F}[f(t)]$$

This may be extended to higher derivatives, so that

$$\mathcal{F}[F''(t)] = i\omega\, \mathcal{F}[f'(t)] = -\omega^2 \tilde{f}(\omega)$$

and so on.

2) Integration:

$$\mathcal{F}\left[\int^{t} f(s)\,ds\right] = \frac{1}{i\omega}\tilde{f}(\omega) + 2\pi c\,\delta(\omega)$$

where the term $2\pi c\,\delta(\omega)$ represents the Fourier transform of the constant of integration associated with the indefinite integral.

3) Scaling:

$$\mathcal{F}[F(at)] = \frac{1}{a}\tilde{f}\left(\frac{\omega}{a}\right)$$

4) Translation:

$$\mathcal{F}[F(t+a)] = e^{ia\omega}\tilde{f}(\omega)$$

5) Exponential multiplication:

$$\mathcal{F}[e^{\alpha t} f(t)] = \tilde{f}(\omega + i\alpha)$$

where α may be real, imaginary or complex.

6) Fourier transform of complex conjugate:

$$\mathcal{F}[F^*(t)] = \tilde{f}^*(-\omega)$$

- Prove the relation $\mathcal{F}[f'(t)] = i\omega \tilde{f}(\omega)$.

Ans:

$$\mathcal{F}[f'(t)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f'(t)\, e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}}\left[e^{-i\omega t} f(t)\right]_{-\infty}^{+\infty} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} i\omega\, e^{-i\omega t} f(t)\, dt$$

$$= i\omega \tilde{f}(\omega).$$

Odd and even functions :

If $F(t)$ is odd or even, then we can define alternative forms of Fourier transform and inverse Fourier transform.

➡ If $f(t)$ is "odd function", i.e. $f(-t) = -f(t)$, we may define the Fourier transform pair for odd functions as :

$$\boxed{\begin{aligned} \tilde{F}_s(\omega) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t)\, \sin(\omega t)\, dt \\ f(t) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \tilde{F}_s(\omega)\, \sin(\omega t)\, d\omega \end{aligned}}$$

← when $f(t)$ is odd function. Also known as "Fourier Sine transform"

➡ If $f(t)$ is "even function", i.e. $f(-t) = f(t)$, then we may define the Fourier transform pair for even functions as :

$$\boxed{\begin{aligned} \tilde{F}_c(\omega) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t)\, \cos(\omega t)\, dt \\ f(t) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \tilde{F}_c(\omega)\, \cos(\omega t)\, d\omega \end{aligned}}$$

← when $f(t)$ is even function. Also known as "Fourier cosine transform".

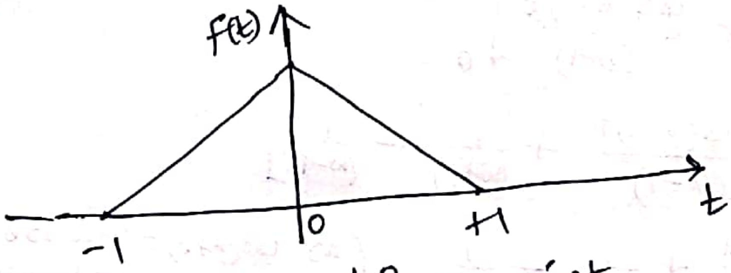
Ex 4: Find the Fourier sine transform of
(i) $\frac{1}{t}$ (ii) $2e^{-3t} + 3e^{-2t}$.

Ans: (i) $\tilde{f}_s(\omega) = \sqrt{\pi/2}$

(ii) $\tilde{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \left[\frac{5\omega^3 + 35\omega}{(9+\omega^2)(4+\omega^2)} \right]$

Ex 5: The "triangle function": Find the Fourier transform of the triangle function defined as:

$$f(t) = \begin{cases} 1-|t| , & |t| \le 1 \\ 0 , & \text{otherwise.} \end{cases}$$



Sol^n:

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{+1} (1-|t|)(\cos\omega t - i\sin\omega t)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{+1} (1-|t|)\cos\omega t\, dt - \frac{i}{\sqrt{2\pi}} \underbrace{\int_{-1}^{+1} (1-|t|)\sin\omega t\, dt}_{0 \text{ (as odd fn.)}}$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{1} (1-t)\cos\omega t\, dt$$

$$= \frac{2}{\sqrt{2\pi}} \left[(1-t)\frac{\sin\omega t}{\omega} \right]_0^1 + \frac{2}{\sqrt{2\pi}} \int_0^1 \frac{\sin\omega t}{\omega}\, dt$$

$$= 0 - \frac{2}{\sqrt{2\pi}} \cdot \frac{1}{\omega^2} \cos\omega t \Big|_0^1$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left(\frac{1-\cos\omega}{\omega^2} \right).$$

Ex6: Given $f(t) = \begin{cases} \sin t & 0<t<\pi \\ 0 & \text{otherwise.} \end{cases}$

Find the Fourier cosine transform of $f(t)$. Then evaluate $\int_0^\infty \frac{\cos \frac{\pi x}{2}}{1-x^2}dx$.

Sol^n:

$$\tilde{F}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\cos\omega t \, dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\pi \sin t \cos\omega t \, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\pi [\sin(\omega+1)t - \sin(\omega-1)t]\, dt$$

(as, $\sin(a+b) = \sin a\cos b + \cos a \sin b$ and $\sin(a-b) = \sin a\cos b - \cos a\sin b$)

$$= \frac{1}{\sqrt{2\pi}}\left[-\frac{\cos(\omega+1)t}{(\omega+1)} + \frac{\cos(\omega-1)t}{(\omega-1)}\right]_0^\pi$$

$$= \frac{1}{\sqrt{2\pi}}\left[-\frac{\cos(\omega+1)\pi}{(\omega+1)} + \frac{\cos(\omega-1)\pi}{(\omega-1)} + \frac{1}{(\omega+1)} - \frac{1}{(\omega-1)}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\cos\omega\pi}{\omega+1} - \frac{\cos\omega\pi}{\omega-1} + \frac{1}{\omega+1} - \frac{1}{\omega-1}\right]$$

(as $\cos(a+b) = \cos a\cos b - \sin a \sin b$ and $\cos(a-b) = \cos a\cos b + \sin a \sin b$)

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{(\omega-1)\cos\omega\pi - (\omega+1)\cos\omega\pi}{(\omega+1)(\omega-1)} + \frac{\omega-1-\omega-1}{(\omega+1)(\omega-1)}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{-2\cos\omega\pi - 2}{\omega^2-1}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{1+\cos\omega\pi}{1-\omega^2}\right]$$

Now taking the inverse Fourier Cosine transform

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \tilde{F}_c(\omega)\cos\omega t\, d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \left[\frac{1+\cos\omega\pi}{1-\omega^2}\right]\cos\omega t\, d\omega = \begin{cases} \sin t, & 0<t<\pi \\ 0, & \text{otherwise.} \end{cases}$$

Putting $t = \pi/2$ on both sides,

$$\Rightarrow \frac{2}{\pi}\int_0^\infty \frac{\cos\omega\pi/2 + \cos\omega\pi\cos\omega\pi/2}{1-\omega^2}\, d\omega = 1.$$

$$\Rightarrow \frac{1}{\pi}\int_0^\infty \frac{2\cos\omega\pi/2 + \left(\cos\frac{3\omega\pi}{2} + \cos\frac{\omega\pi}{2}\right)}{1-\omega^2}\, d\omega = 1$$

$$\Rightarrow \frac{1}{\pi}\int_0^\infty \frac{3\cos\frac{\omega\pi}{2} + \cos\frac{3\omega\pi}{2}}{1-\omega^2}\, d\omega = 1$$

$$= \frac{1}{\pi}\int_0^\infty \frac{2\cos\omega\pi/2}{1-\omega^2}\, d\omega = 1 \quad \left(\text{as } \cos\frac{3\omega\pi}{2} = -\cos\frac{\omega\pi}{2}\right)$$

So,

$$\int_0^\infty \frac{\cos\pi\omega/2}{1-\omega^2}\, d\omega = \frac{\pi}{2}$$

$$\Rightarrow \int_0^\infty \frac{\cos\pi x/2}{1-x^2}\, dx = \pi/2.$$

Ex 7: Solve the integral equation $\int_0^\infty F(t) \cos \omega t \, dt = \begin{cases} 1-\omega, & 0 \le \omega \le 1 \\ 0, & \omega > 1 \end{cases}$.

Hence deduce that, $\int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}$.

Sol^n: Given that, $\int_0^\infty F(t) \cos \omega t \, dt = \begin{cases} 1-\omega, & 0 \le \omega \le 1 \\ 0, & \omega > 1 \end{cases}$

$$\Rightarrow \sqrt{\frac{2}{\pi}} \int_0^\infty F(t) \cos \omega t \, dt = \begin{cases} \sqrt{\frac{2}{\pi}}(1-\omega), & 0 \le \omega \le 1 \\ 0, & \omega > 1 \end{cases}$$

$$\Rightarrow \tilde{F}_c(\omega) = \begin{cases} \sqrt{\frac{2}{\pi}}(1-\omega), & 0 \le \omega \le 1 \\ 0, & \omega > 1 \end{cases}$$

Now taking the inverse Fourier Cosine transform,

$$F(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \tilde{F}_c(\omega) \cos \omega t \, d\omega$$

$$= \frac{2}{\pi} \int_0^1 (1-\omega) \cos \omega t \, d\omega$$

$$= \frac{2}{\pi} \left[(1-\omega) \frac{\sin \omega t}{t} - (-1) \cdot \left(-\frac{\cos \omega t}{t^2} \right) \right]_{\omega=0}^{\omega=1}$$

$$= \frac{2}{\pi} \left[0 - \frac{\cos t}{t^2} + \frac{1}{t^2} \right] = \frac{2}{\pi} \left(\frac{1-\cos t}{t^2} \right) = \frac{2}{\pi} \frac{2\sin^2 t/2}{t^2}$$

$$\Rightarrow F(t) = \frac{4 \sin^2 t/2}{\pi t^2}$$

Now given that, $\int_0^\infty F(t) \cos \omega t \, dt = \begin{cases} 1-\omega, & 0 \le \omega \le 1 \\ 0, & \omega > 1 \end{cases}$

putting $\omega = 0$ on both sides,

$$\int_0^\infty F(t) \, dt = 1$$

$$\Rightarrow \int_0^\infty \frac{4 \sin^2 t/2}{\pi t^2} dt = 1$$

$$\Rightarrow \int_0^\infty \frac{\sin^2 t/2}{t^2} dt = \frac{\pi}{4}, \quad \text{Now putting } t/2 = x; \; 2dx = dt$$

$$\Rightarrow \int_0^\infty \frac{\sin^2 x}{4x^2} \cdot 2dx = \pi/4 \quad \Rightarrow \int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}.$$

Ex 8: Find the Fourier transform of the function

$$f(x) = e^{-a|t|}, \quad -\infty < t < \infty.$$

Sol^n: Given

$$f(x) = \begin{cases} e^{at}, & t < 0 \\ e^{-at}, & t \geq 0 \end{cases}$$

$$\tilde{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{at} e^{-i\omega t}\, dt + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-at} e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(a-i\omega)t}\, dt + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(a+i\omega)t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{(a-i\omega)t}}{a-i\omega} \right]_{-\infty}^{0} + \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-(a+i\omega)t}}{-(a+i\omega)} \right]_{0}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{a-i\omega} + \frac{1}{a+i\omega} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{a+i\omega+a-i\omega}{a^2+\omega^2}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{2a}{a^2+\omega^2}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+\omega^2}$$

Ex 9: Find the Fourier transform of the function $f(t) = e^{-at} H(t)$, $a > 0$, where $H(t)$ represents the unit step function or the Heaviside step function.

Sol^n: From definition of $H(t)$,

$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-at} & \text{for } t \geq 0. \end{cases}$$

$\longrightarrow$ Same as Ex 1. Answer: $\tilde{F}(\omega) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(a+i\omega)}$

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Ex 10. If $\tilde{f}(\omega)$ is Fourier transform of $f(t)$, then show that, $\mathcal{F}[f(at)] = \frac{1}{a}\tilde{f}\left(\frac{\omega}{a}\right)$

Sol^n:
$$\mathcal{F}[f(at)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(at)\, e^{-i\omega t}\, dt$$

putting $at = u$

$\Rightarrow dt = \frac{1}{a} du$

$$\mathcal{F}[f(at)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(u) \cdot \frac{1}{a} \cdot e^{-i\frac{\omega}{a}u}\, du$$

$$= \frac{1}{a} \cdot \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(u) \cdot e^{-i\frac{\omega}{a}\cdot u}\, du$$

$$= \frac{1}{a}\tilde{f}\left(\frac{\omega}{a}\right)$$

Ex. 11. If $\tilde{f}(\omega)$ is Fourier Transform of $f(t)$, then show that, $\mathcal{F}[f(t+a)] = e^{ia\omega}\tilde{f}(\omega)$

Sol^n:
$$\mathcal{F}[f(t+a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t+a)\, e^{-i\omega t}\, dt$$

putting $t + a = u$

$\Rightarrow dt = du$

$$\mathcal{F}[f(t+a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(u) \cdot e^{-i\omega(u-a)} \cdot du$$

$$= e^{ia\omega} \cdot \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(u)\, e^{-i\omega u}\, du = e^{ia\omega}\tilde{f}(\omega).$$

Ex 12 (*)(*) If $\tilde{f}(\omega)$ is Fourier transform of $f(t)$, then show that,

$$\mathcal{F}[f(t)\cos at] = \frac{1}{2}\left[\tilde{f}(\omega+a) + \tilde{f}(\omega-a)\right].$$

known as "Modulation theorem".

Sol^n:
$$\mathcal{F}[f(t)\cos at] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t)\cos at\, e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t) \cdot \frac{1}{2}\left(e^{iat} + e^{-iat}\right) \cdot e^{-i\omega t}\, dt$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t)\, e^{-i(\omega-a)t}\, dt + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t)\, e^{-i(\omega+a)t}\, dt\right]$$

$$= \frac{1}{2}\left[\tilde{f}(\omega+a) + \tilde{f}(\omega-a)\right]$$

Ex13. Find the Fourier tranform of e^{-t^2}. From that result, find Fourier transform of

(i) e^{-at^2}, $a>0$

(ii) $e^{-2(t-3)^2}$

(iii) $e^{-t^2}\cos 2t$.

Solⁿ: $\mathcal{F}[e^{-t^2}] = \tilde{F}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-t^2} e^{-i\omega t}\,dt$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\left[t^2+2\cdot\left(\frac{i\omega}{2}\right)t+\left(\frac{i\omega}{2}\right)^2-\left(\frac{i\omega}{2}\right)^2\right]}dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\left(t+\frac{i\omega}{2}\right)^2-\frac{\omega^2}{4}}\,dt$$

$$= \frac{e^{-\omega^2/4}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\left(t+i\omega/2\right)^2}dt$$

$$= \frac{e^{-\omega^2/4}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-u^2}du \qquad \left(\text{putting } u = t+\frac{i\omega}{2} \Rightarrow du = dt\right)$$

$$= 2\cdot\frac{e^{-\omega^2/4}}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-u^2}du \qquad (\text{as } e^{-u^2} \text{ is even function of } u)$$

$$= 2\cdot\frac{e^{-\omega^2/4}}{\sqrt{2\pi}}\cdot\frac{\sqrt{\pi}}{2} \qquad \left[\text{Recall the formula, } \int_0^\infty e^{-a^2t^2}dt = \frac{\sqrt{\pi}}{2a}\right]$$

$$\Rightarrow \tilde{F}(\omega) = \frac{1}{\sqrt{2}}e^{-\omega^2/4}$$

(i) Now $\mathcal{F}[e^{-at^2}] = \mathcal{F}\left[e^{-(\sqrt{a}t)^2}\right]$

$$= \frac{1}{\sqrt{a}}\tilde{F}\left(\frac{\omega}{\sqrt{a}}\right) \qquad \left[\text{by the change of scale property, i.e. } \mathcal{F}[f(at)] = \frac{1}{a}\tilde{F}\left(\frac{\omega}{a}\right)\right]$$

$$\Rightarrow \mathcal{F}[e^{-at^2}] = \frac{1}{\sqrt{a}}\cdot\frac{1}{\sqrt{2}}e^{-\frac{1}{4}\left(\frac{\omega}{\sqrt{a}}\right)^2}$$

$$= \frac{1}{\sqrt{2a}}e^{-\frac{\omega^2}{4a}}$$

(ii) To find $\mathcal{F}[e^{-2(t-3)^2}]$,

first put $a=2$ in the earlier result

$$\mathcal{F}[e^{-at^2}] = \frac{1}{\sqrt{2a}} e^{-\omega^2/4a}$$

$$\Rightarrow \mathcal{F}[e^{-2t^2}] = \frac{1}{2} e^{-\omega^2/8}$$

Now use the shifting property, i.e. $\mathcal{F}[F(t+a)] = e^{ia\omega} \tilde{F}(\omega)$.
and put $a=-3$. $= e^{ia\omega} \mathcal{F}[F(t)]$

So, $$\mathcal{F}[e^{-2(t-3)^2}] = e^{-3i\omega} \cdot \mathcal{F}[e^{-2t^2}]$$

$$= \frac{1}{2} \cdot e^{-3i\omega} \cdot e^{-\omega^2/8}$$

(iii) $\mathcal{F}[e^{-t^2} \cos 2t]$

We'll use the modulation theorem,

$$\mathcal{F}[F(t) \cos at] = \frac{1}{2}[\tilde{F}(\omega+a) + \tilde{F}(\omega-a)]$$

So, $$\mathcal{F}[e^{-t^2} \cos 2t] = \frac{1}{2}[\tilde{F}(\omega+2) + \tilde{F}(\omega-2)]$$

$\{$ Using. $\mathcal{F}[e^{-t^2}] = \tilde{F}(\omega) = \frac{1}{\sqrt{2}} e^{-\omega^2/4}$ $\}$

$$\Rightarrow \mathcal{F}[e^{-t^2} \cos 2t] = \frac{1}{2}\left[\frac{1}{\sqrt{2}} e^{-(\omega+2)^2/4} + \frac{1}{\sqrt{2}} e^{-(\omega-2)^2/4}\right]$$

Ex 14. Find the Fourier Sine transform of $\frac{d^2F}{dt^2}$. Assume that $F, \frac{dF}{dt}, \frac{d^2F}{dt^2} \to 0$ at $t \to \pm\infty$

Sol^n: $$\mathcal{F}_s\left[\frac{d^2F}{dt^2}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{d^2F}{dt^2} \cdot \sin(\omega t)\, dt$$

$$= \sqrt{\frac{2}{\pi}}\left[\sin(\omega t) \frac{dF}{dt}\Big|_0^\infty - \omega \int_0^\infty \frac{dF}{dt} \cos(\omega t)\, dt\right]$$

$$= 0 - \omega \cdot \sqrt{\frac{2}{\pi}}\left[\cos(\omega t) \cdot F(t)\Big|_0^\infty + \omega \int_0^\infty \sin(\omega t) \cdot F(t)\, dt\right]$$

$$= \sqrt{\frac{2}{\pi}}\, \omega F(0) - \omega^2 \cdot \sqrt{\frac{2}{\pi}} \int_0^\infty F(t) \sin(\omega t)\, dt$$

$$= \sqrt{\frac{2}{\pi}}\, \omega F(0) - \omega^2 \mathcal{F}_s[F(t)].$$

Convolution Theorem :

Convolution of two functions $f(t)$ and $g(t)$ is defined as

$$f(t) * g(t) = \int_{-\infty}^{+\infty} f(u)\, g(t-u)\, du$$

If $\tilde{f}(\omega)$ and $\tilde{g}(\omega)$ are Fourier transforms of $f(t)$ and $g(t)$ respectively, then Convolution theorem for Fourier transforms states that,

$$\boxed{\frac{1}{\sqrt{2\pi}} \mathcal{F}\left[f(t) * g(t)\right] = \mathcal{F}\left[f(t)\right] \cdot \mathcal{F}\left[g(t)\right] = \tilde{f}(\omega) \cdot \tilde{g}(\omega)}$$

Proof:

By definition, $\tilde{f}(\omega) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t}\, dt$

and $\tilde{g}(\omega) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{+\infty} g(t)\, e^{-i\omega t}\, dt$

Now, $f(t) * g(t) = \displaystyle\int_{-\infty}^{+\infty} f(u)\, g(t-u)\, du$

$$\Rightarrow \mathcal{F}\left[f(t) * g(t)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\omega t} \left[\int_{-\infty}^{+\infty} f(u)\, g(t-u)\, du\right] dt$$

Changing the order of integration, we get

$$\Rightarrow \mathcal{F}\left[f(t) * g(t)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(u) \left[\int_{-\infty}^{+\infty} g(t-u)\, e^{-i\omega t}\, dt\right] du$$

putting $t - u = z \Rightarrow dt = dz$, we get

$$\Rightarrow \mathcal{F}\left[f(t) * g(t)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(u) \left[\int_{-\infty}^{+\infty} g(z)\, e^{-i(u+z)\omega}\, dz\right] du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(u)\, e^{-i\omega u} \left[\int_{-\infty}^{+\infty} g(z)\, e^{-i\omega z}\, dz\right] du$$

$$= \int_{-\infty}^{+\infty} f(u)\, e^{-i\omega u}\, \tilde{g}(\omega)\, du$$

$$= \sqrt{2\pi}\, \tilde{g}(\omega) \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(u)\, e^{-i\omega u}\, du$$

$$= \sqrt{2\pi}\, \tilde{g}(\omega) \cdot \tilde{F}(\omega)$$

$$\rightarrow \boxed{\mathcal{F}[f * g] = \sqrt{2\pi}\,.\,\tilde{f}(\omega)\,.\,\tilde{g}(\omega)} \quad \text{— (1)}$$

Hence, the Fourier transform of a convolution $f * g$ is equal to the product of the separate Fourier transforms multiplied by $\sqrt{2\pi}$ $\longrightarrow$ this result is called the "convolution theorem."

It may be proved similarly that the converse is also true, namely that the Fourier transform of the product $f(t).g(t)$ is given by

$$\boxed{\mathcal{F}[\underbrace{f(t).g(t)}_{\text{product}}] = \frac{1}{\sqrt{2\pi}}\, \underbrace{\tilde{f}(\omega) * \tilde{g}(\omega)}_{\text{convolution.}}}$$

Also, if we take inverse Fourier transform of eq^n (1), we can get

$$\boxed{\sqrt{2\pi}\,.\,\mathcal{F}^{-1}\left[\tilde{f}(\omega).\tilde{g}(\omega)\right] = f(t) * g(t)}$$

Fourier transforms in 3-dimensions:

The concept of the Fourier transform can be extended naturally to more than one dimension. For example, consider a function $f(x,y,z)$, [instead of $f(t)$, we take here $f(x,y,z)$ and instead of ω, by conversion we take k_x, k_y and k_z]

In 3-dimension we can define Fourier transform of $f(x,y,z)$ as

$$\circledast\ \mathcal{F}[f(x,y,z)] = \tilde{f}(k_x,k_y,k_z) = \frac{1}{(2\pi)^{3/2}} \iiint f(x,y,z).\, e^{-ik_x x}\, e^{-ik_y y}\, e^{-ik_z z}\, dx\, dy\, dz.$$

and its inverse by as

$$\circledast\ f(x,y,z) = \frac{1}{(2\pi)^{3/2}} \iiint \tilde{f}(k_x,k_y,k_z)\, e^{ik_x x}\, e^{ik_y y}\, e^{ik_z z}\, dk_x\, dk_y\, dk_z.$$

Now, denoting the vector with components k_x, k_y, k_z by $\vec{k}$ and that with components x, y, z by $\vec{r}$, we can write:

$$\tilde{f}(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int f(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}\, d^3\vec{r}$$

and $$f(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int \tilde{f}(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}\, d^3\vec{k}.$$

From these relations, we can deduce that the three-dim. Dirac δ-function as

$$\delta(\vec{r}) = \frac{1}{(2\pi)^3} \int e^{i\vec{k}\cdot\vec{r}}\, d^3\vec{k}$$

• Special case: If the function $f(\vec{r})$ possesses spherical symmetry, i.e. $f(\vec{r}) = f(r)$, then

$$d^3\vec{r} = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$\vec{k}\cdot\vec{r} = kr\cos\theta, \quad \text{where } k = |\vec{k}|$$

$$\tilde{f}(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int f(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}\, d^3\vec{r}$$

$$= \frac{1}{(2\pi)^{3/2}} \int_0^\infty dr \int_0^\pi d\theta \int_0^{2\pi} d\phi\, f(r)\, r^2 \sin\theta\, e^{-ikr\cos\theta}$$

$$= \frac{1}{(2\pi)^{3/2}} \int_0^\infty dr.\ 2\pi f(r)\, r^2 \int_0^\pi d\theta \sin\theta\, e^{-ikr\cos\theta}$$

The integral over θ can be evaluated using

$$\frac{d}{d\theta}\left(e^{-ikr\cos\theta}\right) = ikr\sin\theta\, e^{-ikr\cos\theta}$$

So, $$\tilde{f}(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int_0^\infty dr\ 2\pi f(r)\, r^2 \left[\frac{e^{-ikr\cos\theta}}{ikr}\right]_{\theta=0}^{\theta=\pi}$$

$$= \frac{1}{(2\pi)^{3/2}} \int_0^\infty 4\pi r^2 f(r) \left(\frac{\sin kr}{kr}\right) dr$$

Application of Fourier Transforms to Partial Differential Equations:

Let $F(x,t)$ be a function of two independent variables x and t. We'll be using the Fourier transform formula with respect to the variable x here.

$$\tilde{F}(\omega,t) = \mathcal{F}[f(x,t)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x,t)\, e^{-i\omega x}\, dx$$

Let us revisit the fourier transform of derivatives for this purpose:

Let $f(x,t), \frac{\partial F}{\partial x}, \frac{\partial^2 F}{\partial x^2}, \dots \to 0$ as $x \to \pm\infty$

(A) Then Fourier transforms of $\frac{\partial F}{\partial x}, \frac{\partial^2 F}{\partial x^2}, \dots$ with respect to x are given by:

(1) $$\mathcal{F}\left[\frac{\partial F}{\partial x}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \frac{\partial F}{\partial x}\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\cancel{e^{-i\omega x} F(x,t)}^{\,0}\Big|_{x=-\infty}^{x=+\infty} + i\omega \int_{-\infty}^{+\infty} f(x,t)\, e^{-i\omega x}\, dx \right]$$

$$= i\omega \cdot \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} F(x,t)\, e^{-i\omega x}\, dx$$

$$= i\omega\, \mathcal{F}[F(x,t)] = i\omega\, \tilde{f}(\omega,t)$$

Similarly it can be shown that

$$\mathcal{F}\left[\frac{\partial^2 F}{\partial x^2}\right] = -\omega^2\, \tilde{F}(\omega,t)$$

In general,

$$\boxed{\mathcal{F}\left[\frac{\partial^n F}{\partial x^n}\right] = (i\omega)^n\, \tilde{f}(\omega,t)}$$

② Fourier sine transform of $\dfrac{\partial^2 f}{\partial x^2}$:

$$\mathcal{F}_s\left[\frac{\partial^2 f}{\partial x^2}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 f}{\partial x^2}\sin \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\underbrace{\left(\sin \omega x \frac{\partial f}{\partial x}\right)\Bigg|_{x=0}^{x=\infty}}_{0} - \omega\int_0^\infty \frac{\partial f}{\partial x}\cos \omega x \, dx\right]$$

$$= 0 - \omega.\sqrt{\frac{2}{\pi}}\left[f(x,t).\cos \omega x\right]_0^\infty$$

$$- \omega^2.\sqrt{\frac{2}{\pi}}\int_0^\infty f(x,t)\sin \omega x \, dx$$

$$\Rightarrow \boxed{\mathcal{F}_s\left[\frac{\partial^2 f}{\partial x^2}\right] = \omega\sqrt{\frac{2}{\pi}}\, f(0,t) - \omega^2 \tilde{f}_s(\omega,t)}$$

Fourier "sine"-transform.

③ Similarly, Fourier cosine transform of $\dfrac{\partial^2 f}{\partial x^2}$ is :

$$\mathcal{F}_c\left[\frac{\partial^2 f}{\partial x^2}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 f}{\partial x^2}\cos \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\left(\cos \omega x \frac{\partial f}{\partial x}\right)\Bigg|_0^\infty + \omega\int_0^\infty \frac{\partial f}{\partial x}\sin \omega x \, dx\right]$$

$$= -\sqrt{\frac{2}{\pi}}\left(\frac{\partial f}{\partial x}\right)_{x=0} + \omega.\sqrt{\frac{2}{\pi}}\left[\sin \omega x \, f(x,t)\right]_0^\infty$$

$$- \omega^2\sqrt{\frac{2}{\pi}}\int_0^\infty f(x,t)\cos \omega x \, dx$$

$$\Rightarrow \boxed{\mathcal{F}_c\left[\frac{\partial^2 f}{\partial x^2}\right] = -\sqrt{\frac{2}{\pi}}\left(\frac{\partial f}{\partial x}\right)_{x=0} - \omega^2 \tilde{f}_c(\omega,t)}$$

(B) Fourier transform of $\frac{\partial f}{\partial t}$ with respect to x :

(1) $$\mathcal{F}\left[\frac{\partial f}{\partial t}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\partial f}{\partial t} e^{-i\omega x} dx$$

$$= \frac{d}{dt} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x,t)\, e^{-i\omega x} dx\right]$$

$$\rightarrow \boxed{\mathcal{F}\left[\frac{\partial f}{\partial t}\right] = \frac{d}{dt} \tilde{f}(\omega, t)}$$

Similarly,

(2) $$\boxed{\mathcal{F}_s\left[\frac{\partial f}{\partial t}\right] = \frac{d}{dt} \tilde{f}_s(\omega,t)}$$ ← Fourier Sine transform

(3) $$\boxed{\mathcal{F}_c\left[\frac{\partial f}{\partial t}\right] = \frac{d}{dt} \tilde{f}_c(\omega,t)}$$ ← Fourier Cosine Transform

- Algorithm to solve partial differential equations with boundary values :

1. Apply the suitable transform to the give partial differential eq^n.
   (i) If $-\infty < x < \infty$, apply Fourier transform.
   (ii) If $0 < x < \infty$, we need to check initial value conditions
      [a] If $f(0,t)$ is given $\longrightarrow$ apply Fourier sine transform.
      [b] If $\left(\frac{\partial f}{\partial x}\right)_{x=0}$ is given $\rightarrow$ apply Fourier Cosine transform.

   An ordinary differential eq^n will be obtained after applying these transforms.

2. Solve the ordinary differential eq^n using usual methods.
3. Apply boundary value conditions to evaluate arbitrary constants.
4. Apply inverse transform to get the required expression for $f(x,t)$.

Ex 1: The temperature $f(x,t)$ at any point of an infinite bar satisfies the equation

One dim Heat Equation

$$\frac{\partial f}{\partial t} = k\frac{\partial^2 f}{\partial x^2}, \quad -\infty < x < +\infty, \ t > 0$$

(k is constant)

The initial temperature along the length of the bar is given by

$$f(x,0) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

Determine the expression for $f(x,t)$.

Solution: As the range of x is $(-\infty, +\infty)$, applying Fourier transform to both sides of the given eq^n:

$$\mathcal{F}\left[\frac{\partial f}{\partial t}\right] = k\,\mathcal{F}\left[\frac{\partial^2 f}{\partial x^2}\right]$$

Using the formulas of F.T. for derivative which we derived earlier,

$$\mathcal{F}\left[\frac{\partial f}{\partial t}\right] = \frac{d}{dt}\tilde{f}(\omega,t)$$

$$\mathcal{F}\left[\frac{\partial^2 f}{\partial x^2}\right] = -\omega^2\tilde{f}(\omega,t)$$

So,

$$\frac{d}{dt}\tilde{f}(\omega,t) = -\omega^2 k\,\tilde{f}(\omega,t)$$

$$\Rightarrow \frac{d\tilde{f}}{\tilde{f}} = -\omega^2 k\,dt$$

Solving we can get,

$$\ln(\tilde{f}) = -\omega^2 kt + C \quad (\text{let } C = \ln A)$$

$$\Rightarrow \ln\left(\frac{\tilde{f}}{A}\right) = -\omega^2 kt$$

$$\Rightarrow \tilde{f}(\omega,t) = Ae^{-\omega^2 kt} \quad \text{--- (1)}$$

Putting $t=0$, we get $\tilde{f}(\omega,0) = A$ --- (2)

Now given the initial value

$$f(x,0) = \begin{cases} 1 & \text{For } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

~~Putting t = 0 in eq ①, we get~~

Taking Fourier Transform of $f(x,0)$, we get

$$\tilde{f}(\omega,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x,0)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{+1} e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(-i\omega)} \left[e^{-i\omega x} \right]_{-1}^{+1}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(-i\omega)} \left[e^{-i\omega} - e^{i\omega} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{2i}{i\omega} \left[\frac{e^{i\omega} - e^{-i\omega}}{2i} \right] \quad \left(\text{as } \sin at = \frac{e^{iat} - e^{-iat}}{2i} \right)$$

$$= \frac{2}{\sqrt{2\pi}} \cdot \frac{\sin\omega}{\omega} = \sqrt{\frac{2}{\pi}}\, \frac{\sin\omega}{\omega} \quad \text{——— ③}$$

Equating eq^n ② & ③, we get

$$A = \sqrt{\frac{2}{\pi}}\, \frac{\sin\omega}{\omega}.$$

Now use this value in eq^n ①, we can get,

$$\tilde{f}(\omega,t) = A\, e^{-\omega^2 kt}$$

$$\tilde{f}(\omega,t) = \sqrt{\frac{2}{\pi}}\, \frac{\sin\omega}{\omega}\, e^{-\omega^2 kt}$$

Taking inverse fourier transform:

$$f(x,t) = \mathcal{F}^{-1}\left[\tilde{f}(\omega,t)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{f}(\omega,t)\, e^{i\omega x}\, d\omega$$

$$\Rightarrow f(x,t) = \frac{1}{\sqrt{2\pi}} \cdot \sqrt{\frac{2}{\pi}} \int_{-\infty}^{+\infty} \frac{\sin\omega}{\omega}\, e^{-\omega^2 kt}\, \underline{e^{i\omega x}}\, d\omega$$

$$= \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\sin\omega}{\omega}\, e^{-\omega^2 kt}\, \underline{(\cos\omega x + i \sin\omega x)}\, d\omega$$

$$\Rightarrow \boxed{f(x,t) = \frac{2}{\pi}\int_0^\infty e^{-\omega^2 kt}\left(\frac{\sin\omega \cos\omega x}{\omega}\right) d\omega}$$

$\left(\text{as } \frac{\sin\omega \sin\omega x}{\omega} \text{ is odd function of } \omega\right)$

Solution for $f(x,t)$

Ex 2: Using Fourier transform, solve the eq^n

$$\frac{\partial f}{\partial t} = k \frac{\partial^2 f}{\partial x^2}, \quad 0 < x < \infty, \ t > 0$$

subject to conditions;

i) $f(0,t) = 0, \quad t > 0$

ii) $f(x,0) = e^{-x}, \quad x > 0$

iii) f and $\frac{\partial f}{\partial x}$ both tend to zero as $x \to \pm\infty$.

Solution: As range of x is $(0,\infty)$, and value of $f(0,t)$ is given, we shall apply Fourier sine transformation, Applying Fourier sine transformation to both sides of the given PDE, we get:

$$\mathcal{F}_s\left[\frac{\partial f}{\partial t}\right] = k\,\mathcal{F}_s\left[\frac{\partial^2 f}{\partial x^2}\right]$$

Recall the formulas for Fourier sine transform of derivatives,

$$\mathcal{F}_s\left[\frac{\partial f}{\partial t}\right] = \frac{d}{dt}\tilde{F}_s(\omega,t)$$

$$\mathcal{F}_s\left[\frac{\partial^2 f}{\partial x^2}\right] = \omega\sqrt{\frac{2}{\pi}}\,f(0,t) - \omega^2\tilde{F}_s(\omega,t)$$

$$\Rightarrow \frac{d}{dt}\tilde{F}_s(\omega,t) = k\omega\sqrt{\frac{2}{\pi}}\,f(0,t) - \omega^2 k\,\tilde{F}_s(\omega,t)$$

$$\Rightarrow \frac{d}{dt}\tilde{F}_s(\omega,t) = -\omega^2 k\,\tilde{F}_s(\omega,t) \quad \left[\text{as } f(0,t)=0 \text{ using initial condition}\right]$$

$$\Rightarrow \frac{d\tilde{F}_s}{\tilde{F}_s} = -w^2 k\, dt$$

Solving this, we get,

$$\ln \tilde{F}_s = -w^2 kt + \ln A$$

$$\Rightarrow \tilde{F}_s(w,t) = A e^{-w^2 kt} \quad \text{——— (1)}$$

Putting $t=0$ on both sides,

$$\tilde{F}_s(w,0) = A \quad \text{——— (2)}$$

Now given the initial condition, $f(x,0) = e^{-x}$, $x>0$

taking Fourier sine transformation ~~both sides~~ of $f(x,0)$, we get:

$$\tilde{f}_s(w,0) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x,0) \sin wx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin wx \, dx$$

$$\Rightarrow \tilde{F}_s(w,0) = \sqrt{\frac{2}{\pi}} \frac{w}{1+w^2} \quad \text{——— (3)}$$

$\left(\text{as remember, } \int_0^\infty e^{-at} \sin wt \, dt = \frac{w}{a^2+w^2}\right)$

Equating eq^ns (2) & (3), we get

$$A = \sqrt{\frac{2}{\pi}} \frac{w}{1+w^2}$$

Using this into eq^n (1), we get

$$\tilde{f}_s(w,t) = \sqrt{\frac{2}{\pi}} \frac{w}{1+w^2} e^{-w^2 kt}$$

Taking Inverse Fourier sine transformation, we get

$$f(x,t) = \mathcal{F}_s^{-1}[\tilde{F}_s(w,t)] = \sqrt{\frac{2}{\pi}} \int_0^\infty \tilde{F}_s(w,t) \sin(wx)\, dw$$

$$\Rightarrow \boxed{F(x,t) = \frac{2}{\pi} \int_0^\infty \frac{w}{1+w^2} e^{-w^2 kt} \sin(wx)\, dw} \leftarrow \text{Solution}$$

Ex 3: An infinite string is initially at rest and its initial displacement is given by $y(x)$, $-\infty < x < +\infty$. Determine the displacement $f(x,t)$ of the string. The eq^n of the vibrating string is given by:

One-dim. wave eq^n

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}.$$

Sol^n: Initial conditions are:

i) $\left.\frac{\partial f}{\partial t}\right|_{t=0} = 0$

ii) $f(x,0) = y(x)$

Taking Fourier transform on both side of the eq^n (as x is in range $(-\infty, +\infty)$):

$$\mathcal{F}\left[\frac{\partial^2 f}{\partial t^2}\right] = c^2 \mathcal{F}\left[\frac{\partial^2 f}{\partial x^2}\right]$$

$$\Rightarrow \frac{d^2}{dt^2}\tilde{f}(\omega,t) = -c^2\omega^2 \tilde{f}(\omega,t)$$

$$\Rightarrow \frac{d^2\tilde{f}}{dt^2} + c^2\omega^2\tilde{f} = 0$$

Solution of this eq^n is:

$$\tilde{f}(\omega,t) = A\cos(c\omega t) + B\sin(c\omega t) \quad \text{—— (1)}$$

Putting $t=0$ on both sides,

$$\tilde{f}(\omega,0) = A \quad \text{—— (2)}$$

Given the initial condition $f(x,0) = y(x)$, taking Fourier transform of $f(x,0)$, we get

$$\tilde{f}(\omega,0) = \mathcal{F}[f(x,0)]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x,0)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} y(x)\, e^{-i\omega x}\, dx$$

$$\Rightarrow \tilde{f}(\omega,0) = \tilde{y}(\omega) \quad \text{—— (3)}$$

Equating eq^{n}s (2) & (3), we get

$$A = \tilde{y}(\omega)$$

Putting this value into eq^n (1), we get

$$\tilde{F}(\omega,t) = \tilde{y}(\omega) \cos(c\omega t) + B \sin(c\omega t)$$

Taking Inverse Fourier transform, we get

$$f(x,t) = \mathcal{F}^{-1}[\tilde{F}(\omega,t)]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{F}(\omega,t)\, e^{i\omega x}\, d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{y}(\omega) \cos(c\omega t)\, e^{i\omega x}\, d\omega + \frac{1}{\sqrt{2\pi}} B \int_{-\infty}^{+\infty} \sin(c\omega t)\, e^{i\omega x}\, d\omega \quad \text{--- (4)}$$

Now taking the derivative w.r.t. t on both sides,

$$\frac{df}{dt} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{y}(\omega) [-c\omega \sin(c\omega t)]\, e^{i\omega x}\, d\omega + \frac{1}{\sqrt{2\pi}} B \int_{-\infty}^{+\infty} (c\omega) \cos(c\omega t) e^{i\omega x}\, d\omega$$

Given the initial condition $\left.\frac{df}{dt}\right|_{t=0} = 0$

We can get,

$$\left.\frac{df}{dt}\right|_{t=0} = 0 + \frac{1}{\sqrt{2\pi}} \cdot B \int_{-\infty}^{+\infty} c\omega\, e^{i\omega x}\, d\omega = 0$$

$$\Rightarrow B = 0$$

So, putting $B = 0$ in eq^n (4), we get

$$\boxed{f(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{y}(\omega) \cos(c\omega t)\, e^{i\omega x}\, d\omega}$$

solution.