Symmetries, Invariance and the Hamiltonian

Motivation:

We saw the advantages of Lagrangian mechanics ~~for~~ over the Newtonian. Newtonian mechanics is given in terms of force vectors on a system, whereas the Lagrangian method is expressed in terms of the scalar kinetic and potential energies of the system and it allows us to ignore constraint forces. In addition, the use of Lagrange multipliers or generalized forces can allow the Lagrangian approach to determine the constraint forces if we want. However, the eq^n's of motion derived from either Newton's Laws or Lagrangian dynamics, can be non-trivial to solve mathematically. It is necessary to integrate second-order eq^n's, which for n degrees of freedom, imply 2n constant of integration.

In this chapter, we'll see that, even when the eq^n's of motion cannot be solved easily, it is possible to derive important physical principles regarding the first-order integrals of motion of the system directly from the Lagrange eq^n, as well as deriving the underlying symmetries plus invariance. This property is contained in Noether's theorem which states that conservation laws are associated with differentiable symmetries of a physical system.

Generalized momentum:

Consider a holonomic system of N particles under the influence of conservative forces that depend on position q_j but not velocity $\dot{q}_j$, that is the potential is velocity independent. Then for the x-coordinate of particle i,

$$\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} - \frac{\partial U}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i}$$

$$= \frac{\partial}{\partial \dot{x}_i} \sum_{i=1}^{N} \frac{1}{2} m_i \left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2\right)$$

$$= m_i \dot{x}_i = p_{i,x}$$

Thus for a holonomic, conservative, velocity-independent potential we have

$$\boxed{\frac{\partial L}{\partial \dot{x}_i} = p_{i,x}}$$ ← x-component of the linear momentum for the i^{th} particle.

This result suggests an obvious extension to the concept of momentum to generalized coordinates.

The "generalized momentum" associated with the coordinate q_j is defined to be

$$\boxed{\frac{\partial L}{\partial \dot{q}_j} = p_j}$$ ← also called "conjugate momentum" or "canonical momentum" to q_j.

Note that, if q_j is not a spacial coordinate, then p_j is the generalized momentum, not the kinetic linear momentum. For example, if q_j is an angle, then p_j will be angular momentum.

Invariant transformation and Noether's Theorem:

For any set of generalized coordinates q_j, a reversible point transformation can define another set of coordinates q'_j such that,

$$q'_j = q'_j(q_1, q_2, q_3, \ldots, q_n, t)$$

The new set of generalized coordinates satisfies Lagrange's eq^n of motion with the new Lagrangian,

$$L(q', \dot{q}', t) = L(q, \dot{q}, t)$$

The Lagrangian itself is a scalar with units of energy which doesn't change when the coordinate representation is changed. So, $L(q', \dot{q}', t)$ can be derived from $L(q, \dot{q}, t)$ just by substituting the inverse relation $q_i = q_i(q'_1, q'_2, \ldots, q'_n, t)$ into $L(q, \dot{q}, t)$.

That is the value of the Lagrangian L is independent of which coordinate representation is used.

But the explicit form of eq^n's of motion for the new variables can be different from that with the old variables. Example — transformation from Cartesian to spherical coordinates.

For holonomic and conservative systems, the Euler-Lagrange eq^n is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

$$\Rightarrow \frac{d}{dt} p_j = \frac{\partial L}{\partial q_j}$$

$$\Rightarrow \dot{p}_j = \frac{\partial L}{\partial q_j}$$

Thus if the Lagrangian L doesn't contain q_j explicitly, the Lagrangian is invariant to a linear translation.

i.e. $\dot{p}_j = \frac{dp_j}{dt} = 0$, p_j being a constant of motion.

This relation is called "Noether's theorem" which states

"For each symmetry of the Lagrangian, there is a conserved quantity".

- Rotational counterpart: Similarly, if the Lagrangian is not an explicit function of θ, then. $\frac{\partial L}{\partial \theta} = 0$, so that

$$\dot{p}_\theta = \frac{\partial L}{\partial \theta} = 0, \text{ then } p_\theta \text{ is a constant of motion.}$$

Noether's theorem for rotation thus states that, if the Lagrangian is rotationally invariant about some axis, then the ~~corresponding~~ component of the angular momentum along that axis is conserved.

- Cyclic coordinate: A "cyclic coordinate" is one that doesn't explicitly appear in the Lagrangian.

(In Hamiltonian mechanics, a cyclic coordinate often is called an "ignorable coordinate".)

In other words, a cyclic coordinate q_k is the one for which $\frac{\partial L}{\partial q_k} = 0$, thus, from E-L eq^n,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \dot{p}_k = 0$$

That is, p_k is a constant of motion if the conjugate coordinate q_k is cyclic. — Same as Noether's theorem.

Generalized Energy and the Hamiltonian function:

Consider the time derivative of the Lagrangian, plus the fact that time is the independent variable in the Lagrangian. The total time derivative of $L(q, \dot{q}, t)$ is

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j} \dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j + \frac{\partial L}{\partial t}$$

The Euler-Lagrange eqn is given as.

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

[It can also be proven using the generalized form of E-L eqn i.e. considering Lagrange multipliers and generalized forces.]

So,

$$\frac{dL}{dt} = \sum_j \dot{q}_j \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} + \sum_j \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j + \frac{\partial L}{\partial t}$$

$$= \sum_j \frac{d}{dt} \left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \right) + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{d}{dt} \left[\sum_j \left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \right) - L \right] = - \frac{\partial L}{\partial t} \qquad \text{(1)}$$

Define "Jacobi's Generalized Energy" $h(q, \dot{q}, t)$ as

$$h(q, \dot{q}, t) \equiv \sum_j \left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \right) - L(q, \dot{q}, t)$$

also knows as "Jacobi's energy integral".

Now define the "Hamiltonian Function" to express this generalized energy in terms of the conjugate variables (q_j, p_j):

$$\boxed{\begin{aligned} H(q, p, t) \equiv h(q, \dot{q}, t) &\equiv \sum_j \left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \right) - L(q, \dot{q}, t) \\ &= \sum_j (\dot{q}_j p_j) - L(q, \dot{q}, t) \end{aligned}} \qquad \text{(11)}$$

- **Conservation of generalized energy:**

Combining eq^n's 1 & 11, we get

$$\frac{dH(q,p,t)}{dt} = \frac{dh(q,\dot{q},t)}{dt} = -\frac{\partial L(q,\dot{q},t)}{\partial t}$$

$$\Rightarrow \boxed{\frac{dH}{dt} = -\frac{\partial L}{\partial t}}$$

If the Lagrangian is not an explicit function of time ~~then~~ ~~the Hami~~ (and if the external forces are zero ← this is needed for the full generalized E-L eq^n), then the Hamiltonian is a constant of motion.

- **Generalized energy and total energy:**

Let us first write down the kinetic energy in generalized coordinates.

In terms of fixed rectangular coordinates, the kinetic energy for N bodies, each having ~~3~~ 3 degrees of freedom, is expressed as:

$$T = \frac{1}{2}\sum_{\alpha=1}^{N}\sum_{i=1}^{3} m_\alpha \dot{x}_{\alpha,i}^2$$

Now the transformation to generalized coordinates is

$$x_{\alpha,i} = x_{\alpha,i}(q_j, t)$$

$$\text{So,}\quad \dot{x}_{\alpha,i} = \sum_{j=1}^{s} \frac{\partial x_{\alpha,i}}{\partial q_j}\dot{q}_j + \frac{\partial x_{\alpha,i}}{\partial t}$$

Taking the square of $\dot{x}_{\alpha,i}$, we can write the generalized kinetic energy as:

$$T(q,\dot{q},t) = \sum_\alpha \sum_{i,j,k} \frac{1}{2} m_\alpha \frac{\partial x_{\alpha,i}}{\partial q_j}\frac{\partial x_{\alpha,i}}{\partial q_k}\dot{q}_j\dot{q}_k + \sum_\alpha\sum_{i,j} m_\alpha \frac{\partial x_{\alpha,i}}{\partial q_j}\frac{\partial x_{\alpha,i}}{\partial t}\dot{q}_j + \sum_\alpha\sum_i \frac{1}{2} m_\alpha \left(\frac{\partial x_{\alpha,i}}{\partial t}\right)^2.$$

$\Rightarrow T(q,\dot{q},t) = T_2(q,\dot{q},t) + T_1(q,\dot{q},t) + T_0(q,t)$

where, $$T_2(q,\dot{q},t) = \sum_\alpha \sum_{i,j,k} \frac{1}{2} m_\alpha \frac{\partial x_{\alpha,i}}{\partial q_j} \frac{\partial x_{\alpha,i}}{\partial q_k} \dot{q}_j \dot{q}_k$$
$$= \sum_{j,k} a_{jk} \dot{q}_j \dot{q}_k$$

$$T_1(q,\dot{q},t) = \sum_\alpha \sum_{i,j} m_\alpha \frac{\partial x_{\alpha,i}}{\partial q_j} \frac{\partial x_{\alpha,i}}{\partial t} \dot{q}_j = \sum_{jk} b_j \dot{q}_j$$

$$T_0(q,t) = \sum_\alpha \sum_i \frac{1}{2} m_\alpha \left(\frac{\partial x_{\alpha,i}}{\partial t}\right)^2$$

$\Rightarrow$ When the transformed system is "scleronomic", time doesn't appear explicitly in the transformation eq"s to generalized coordinates since $\frac{\partial x_{\alpha,i}}{\partial t} = 0$, then $T_1 = T_0 = 0$.

The kinetic energy reduces to

$$T(q,\dot{q},t) = T_2(q,\dot{q},t)$$

$\Rightarrow$ Now if the potential energy U doesn't depend explicitly on velocities $\dot{q}$ or time t, then

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial (T-U)}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$$

For such system, Hamiltonian can be written as

$$H(q,p,t) = \sum_j (\dot{q}_j p_j) - L(q,\dot{q},t)$$
$$= \sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} - (T-U)$$

Now, $$T(q,\dot{q},t) = T_2(q,\dot{q},t) = \sum_{j,k} a_{jk} \dot{q}_j \dot{q}_k$$
$$\frac{\partial T}{\partial \dot{q}_l} = \frac{\partial T_2}{\partial \dot{q}_l} = \sum_k a_{lk} \dot{q}_k + \sum_j a_{jl} \dot{q}_j$$

so, $$\sum_l \dot{q}_l \frac{\partial T_2}{\partial \dot{q}_l} = \sum_{k,l} a_{lk} \dot{q}_k \dot{q}_l + \sum_{j,l} a_{jl} \dot{q}_j \dot{q}_l$$

$$\Rightarrow \sum_l \dot{q}_l \frac{\partial T_2}{\partial \dot{q}_l} = 2\sum_{k,l} a_{lk}\dot{q}_k\dot{q}_l = 2T_2$$

(as k, l, j all are dummy indices)

So, $$\sum_j \dot{q}_j \frac{\partial T_2}{\partial \dot{q}_j} = 2T_2$$ (as l, j are dummy indices and summed over)

Thus,

$$H(q,p,t) = \sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} - (T-U)$$

$$= 2T_2 - T_2 + U$$

$$\Rightarrow \boxed{H = T_2 + U}$$

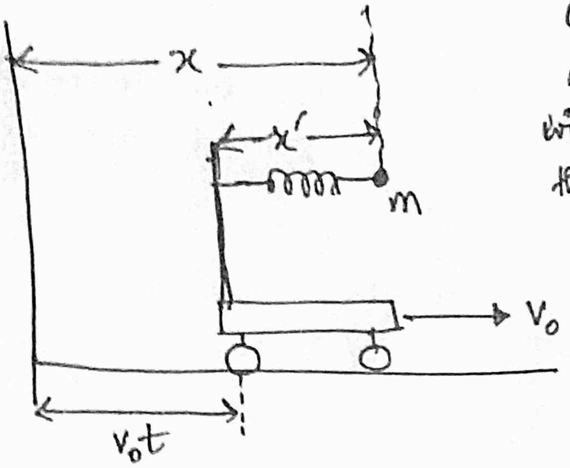
Note that, Hamiltonian doesn't equal to the total energy E. However, in the special case when the transformation is scleronomic, i.e. $T_1 = T_0 = 0$ and if the potential energy U doesn't explicitly depend on $\dot{q}_j$, then the generalized energy or Hamiltonian equals the total energy,

i.e. $$\boxed{H = T + U = E}$$

This is equivalent to saying that, if the constraints for the system are time or generalized coordinates independent, then H = E.

Hamiltonian time behavior	Constraints and coordinate transformation	
	Time independent	Time dependent
$\frac{dH}{dt} = -\frac{\partial L}{\partial t} = 0$	H is conserved, H = E	H is conserved, $H \neq E$
$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \neq 0$	H not conserved, H = E	H not conserved, $H \neq E$

• **Example :** Linear harmonic oscillator on a cart moving at constant velocity



Consider a linear harmonic oscillator on a cart moving with constant velocity v_0 in the x-direction.

Let the "Laboratory frame" be the unprimed and the "cart frame" be the primed frame.

Assume that $x = x'$ at $t = 0$, then

$$x' = x - v_0 t$$
$$\dot{x}' = \dot{x} - v_0$$
$$\ddot{x}' = \ddot{x}$$

Potential energy of the harmonic oscillator

$$U = \frac{1}{2}kx'^2 = \frac{1}{2}k(x - v_0 t)^2$$

(i) Laboratory frame: The Lagrangian is

$$L(x, \dot{x}, t) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}k(x - v_0 t)^2$$

E-L eq^n gives,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$

$$\Rightarrow \quad m\ddot{x} = -k(x - v_0 t)$$

Generalized momentum $\quad p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$

The Hamiltonian is

$$H(x, p, t) = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L$$

$$= m\dot{x}^2 - \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k(x - v_0 t)^2$$

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k(x - v_0 t)^2$$

$$= \frac{p^2}{2m} + \frac{1}{2}k(x - v_0 t)^2$$

Thus, the Hamiltonian is the sum of the kinetic and potential energies, i.e. equals the total energy of the system.

But it is not conserved since L and H are both explicit functions of time, i.e. $\frac{dH}{dt} = -\frac{\partial L}{\partial t} \neq 0$

(ii) Cart frame:

Now transform the Lagrangian to the primed coordinates in the moving frame of reference, which also is an inertial frame.

$$L(x', \dot{x}', t) = \frac{1}{2}m(\dot{x}' + v_0)^2 - \frac{1}{2}kx'^2$$

$$= \frac{1}{2}m(\dot{x}'^2 + 2\dot{x}'v_0 + v_0^2) - \frac{1}{2}kx'^2$$

E-L eqⁿ of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}'} - \frac{\partial L}{\partial x'} = 0$$

$$\Rightarrow \quad m\ddot{x}' = -kx'$$

This implies that an observer on the cart will observe simple harmonic ~~oscill~~ motion as it is expected from the principle of equivalence in Galilean relativity.

Generalized momentum in primed frame coordinates will be

$$p' = \frac{\partial L}{\partial \dot{x}'} = m\dot{x}' + mv_0 = m(\dot{x}' + v_0)$$

Hamiltonian in terms of primed coordinate,

$$H(x', p', t) = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L$$

$$= \dot{x}'\frac{\partial L}{\partial \dot{x}'} - L$$

$$= m\dot{x}'^2 + m\dot{x}'v_0 - \frac{1}{2}m\dot{x}'^2 - m\dot{x}'v_0 - \frac{1}{2}mv_0^2 + \frac{1}{2}kx'^2$$

$$= \frac{1}{2}m\dot{x}'^2 - \frac{1}{2}mv_0^2 + \frac{1}{2}kx'^2$$

$$= \frac{(p' - mv_0)^2}{2m} + \frac{1}{2}kx'^2 - \frac{1}{2}mv_0^2$$

Thus, ~~the~~ both the Lagrangian and Hamiltonian expressed in terms of the coordinates in the cart (primed) frame of reference are not explicitly time dependent, so

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} = 0 \longrightarrow \text{therefore } H \text{ is conserved.}$$

However, the cart-frame Hamiltonian does not equal the total energy, since the coordinate transformation is time dependent.

Hamiltonian for cyclic coordinates:

For cyclic coordinates q_k for which $\frac{\partial L}{\partial q_k} = 0$ (ignoring the external and Lagrange multiplier terms), we have

$$\dot{p}_k = \frac{\partial L}{\partial q_k} = -\frac{\partial H}{\partial q_k} = 0$$

That is, a cyclic coordinate has a constant corresponding momentum p_k for the Hamiltonian. Conversely, if a generalized coordinate doesn't occur in the Hamiltonian, then the corresponding generalized momentum is conserved.

For example, if the Lagrangian, or Hamiltonian do not depend on a linear coordinate x, then p_x is conserved; similarly for θ, p_θ is conserved.

An extension of this principle can give that, the Hamiltonian equals the total energy if the transformation to generalized coordinate is time independent and the potential energy is velocity independent.

✱✱✱ Advantage of using Hamiltonian formulation:

Consider the Lagrangian has one cyclic coordinate variable, say q_n. So, L doesn't depend on q_n,

$$L = L(q_1, \dots, q_{n-1}, \dot{q}_1, \dots, \dot{q}_n, t)$$

↳ The Lagrangian still contains n generalized velocities, thus one still has to deal with 'n' degrees of freedom, even though one degree of freedom q_n is cyclic.

However, in the Hamiltonian formulation, only $(n-1)$ degrees of freedom are required, since the momentum ~~for~~ for the cyclic coordinate is constant, $p_n = \alpha$.

$$H = H(q_1, \dots, q_{n-1}, p_1, \dots, p_{n-1}, \alpha, t)$$

↳ Includes only $(n-1)$ degrees of freedom.

Hamiltonian mechanics can significantly reduce the dimension of the problem when the system involves several cyclic coordinates.

Legendre Transformation between Lagrangian and Hamiltonian

Hamiltonian mechanics can be derived directly from Lagrange mechanics by considering the "Legendre transformation" between the conjugate variables $(q, \dot{q}, t)$ and (q, p, t). Such derivation shows that Hamiltonian mechanics is based on the same variational principle (or the d'Alembert's principle + Hamilton's principle) as those used to derive Lagrangian mechanics.

• Legendre transformation:

Consider transformation between two functions $F(u, w)$ and $G(v, w)$, where u, v are active variables and w is passive variable, ~~[illegible]~~ related by the functional form:

$$v = \frac{\partial}{\partial u} F(u, w)$$

[~~[illegible]~~ we can prove all the findings here considering a general vector form as well, i.e. $\vec{v} = \nabla_u F(\vec{u}, \vec{w})$.]

The Legendre transformation states that, the inverse formula can always be written as a first-order derivative

$$u = \frac{\partial}{\partial v} G(v, w)$$

Thus the function $G(v, w)$ and $F(u, w)$ are related by the symmetric relation:

$$\boxed{G(v, w) + F(u, w) = uv}$$

Furthermore, the first-order derivative with respect to ~~[illegible]~~ the passive variable w is related ~~[illegible]~~ by:

$$\boxed{\frac{\partial}{\partial w} F(u, w) = -\frac{\partial}{\partial w} G(v, w)}$$

← taking derivative w.r.t. w.

$F(u,w)$ & $G(v,w)$ ~~are~~ each is said to be the Legendre transform of the other.

- Now we'll use the Legendre transformation to relate Lagrangian and Hamiltonian.

Identify the active variable v with p, u with $\dot{q}$, and the passive variable w with q and t, and the corresponding functions

$$F(u,w) = L(q,\dot{q},t)$$

and $G(v,w) = H(q,p,t)$.

As the generalize momentum corresponds to

$$p = \frac{\partial}{\partial \dot{q}} L(q,\dot{q},t), \quad \left[\text{as we had } v = \frac{\partial}{\partial u} F(u,w)\right],$$

the Legendre transform states that the transformed variable $\dot{q}$ is given by the relation:

$$\dot{q} = \frac{\partial}{\partial p} H(q,p,t)$$

Since the functions $L(q,\dot{q},t)$ and $H(q,p,t)$ are the Legendre transform of each other, they satisfy the relation:

$$H(q,p,t) + L(q,\dot{q},t) = \dot{q}p$$

Considering all the generalized coordinates, we have

$$H(q,p,t) + L(q,\dot{q},t) = \sum_j \dot{q}_j p_j$$

$$\Rightarrow H(q,p,t) = \sum_j \dot{q}_j p_j - L(q,\dot{q},t)$$

identical to what we define our Hamiltonian as generalized energy earlier.

Taking the derivatives with respect to passive variables q & t, and writing for all the generalized coordinates, we get:

$$\boxed{\begin{aligned} \frac{\partial L(\dot{q},q,t)}{\partial q_j} &= -\frac{\partial H(q,p,t)}{\partial q_j} \\ \frac{\partial L(\dot{q},q,t)}{\partial t} &= -\frac{\partial H(q,p,t)}{\partial t} \end{aligned}}$$

Hamilton's equation of motion:

(i) If the system is holonomic and conservative,

$$\dot{q}_j = \frac{\partial H(q,p,t)}{\partial p_j}$$

$$\dot{p}_j = -\frac{\partial H(q,p,t)}{\partial q_j} \quad \Big(\text{as } p_j = \frac{\partial L}{\partial \dot{q}_j}$$

$$\Rightarrow \dot{p}_j = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_j} \quad [\text{using E-L eq}^n]$$

$$\Rightarrow \dot{p}_j = -\frac{\partial H}{\partial q_j} \Big)$$

$$\frac{dH(q,p,t)}{dt} = \frac{\partial H(q,p,t)}{\partial t} = -\frac{\partial L(q,\dot{q},t)}{\partial t}$$

(ii) For more general case, i.e. taking Lagrange multipliers and generalized forces, we can get:

$$\dot{q}_j = \frac{\partial H(q,p,t)}{\partial p_j}$$

$$\dot{p}_j = -\frac{\partial H(q,p,t)}{\partial q_j} + \left[\sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j} + Q_j^{EXC}\right]$$

$$\frac{dH(q,p,t)}{dt} = \sum_j \left(\left[\sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j} + Q_j^{EXC}\right] \dot{q}_j \right) - \frac{\partial L(q,\dot{q},t)}{\partial t}$$

Applications of Hamiltonian Dynamics:

- **Steps:** Formally the Hamiltonian is constructed from the Lagrangian.
  1. Select independent set of generalized coordinates q_j
  2. Construct Lagrangian $L(q_j, \dot{q}_j, t)$
  3. Derive the conjugate generalized momenta via $p_j = \dfrac{\partial L}{\partial \dot{q}_j}$
  4. Derive $H = \sum_j \dot{q}_j p_j - L$
  5. Derive $\dot{q}_j = \dfrac{\partial H}{\partial p_j}$

and $$\dot{p}_j = -\frac{\partial H(q,p,t)}{\partial q_j} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j} + Q_j^{EXC}$$

Fortunately, the above lengthy procedure often can be bypassed for "conservative systems", i.e. if the following conditions are satisfied:

(i) $L = T(\dot{q}) - U(q)$, that is $U(q)$ is independent of the velocity $\dot{q}$.

(ii) the generalized coordinates are time independent,

then it is possible to use the fact that $\boxed{H = T + U = E}$

And if the system is holonomic, we can simply use
$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}.$$

Example 1: Motion in a uniform gravitational field:

Consider a mass m in a uniform gravitational field acting in the $-\hat{z}$ direction.

The Lagrangian $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$

The generalized momenta are
$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}, \quad p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

The Hamiltonian is
$$H = \sum_j p_j \dot{q}_j - L = p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - L$$
$$= \frac{p_x^2}{m} + \frac{p_y^2}{m} + \frac{p_z^2}{m} - \frac{1}{2}\left(\frac{p_x^2}{m} + \frac{p_y^2}{m} + \frac{p_z^2}{m}\right) + mgz$$
$$= \frac{1}{2}\left(\frac{p_x^2}{m} + \frac{p_y^2}{m} + \frac{p_z^2}{m}\right) + mgz = \underbrace{T + U}_{\text{(Conservative System)}}$$

Note that, the Lagrangian is not explicitly time dependent, thus the Hamiltonian is a constant of motion.

Hamilton's eq^{n}s give:

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \qquad \dot{p}_x = -\frac{\partial H}{\partial x} = 0$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m} \qquad \dot{p}_y = -\frac{\partial H}{\partial y} = 0$$

$\rightarrow$ p_x, p_y are Constant of motion

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \qquad \dot{p}_z = -\frac{\partial H}{\partial z} = -mg$$

Combining these gives, $\ddot{x} = 0, \ddot{y} = 0, \ddot{z} = -mg$.

Example 2: One-dimensional harmonic oscillator:

Consider a mass m subject to a linear restoring force with constant k.

$$L = T - U = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$$

Generalized momentum, $p_x = \dfrac{\partial L}{\partial \dot{x}} = m\dot{x}$

$$H = \sum_j p_j \dot{q}_j - L = p_x \dot{x} - L$$

$$= \frac{1}{2}\frac{p_x^2}{m} + \frac{1}{2}kx^2 = T + U$$

(Conservative system)

Hamilton's eq^{n}s of motion:

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \qquad \dot{p}_x = -\frac{\partial H}{\partial x} = -\frac{\partial U}{\partial x}$$

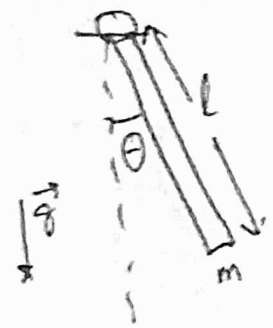
$$\Rightarrow p_x = m\dot{x} \qquad = -kx$$

Combining these gives,

$$\ddot{x} + \frac{k}{m} x = 0$$

which is the eq^n of motion for the harmonic oscillator.

Example 3: Plane pendulum:



There is only one generalized coordinate, θ.

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta$$

The conjugate momentum of θ is

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta} \rightarrow \text{angular momentum about the pivot point.}$$

Hamiltonian is

$$H = \sum_j p_j \dot{q}_j - L = p_\theta \dot{\theta} - L$$

$$= \frac{1}{2} m l^2 \dot{\theta}^2 - mgl\cos\theta$$

$$= \frac{p_\theta^2}{2ml^2} - mgl\cos\theta . = T + U$$

(conservative system)

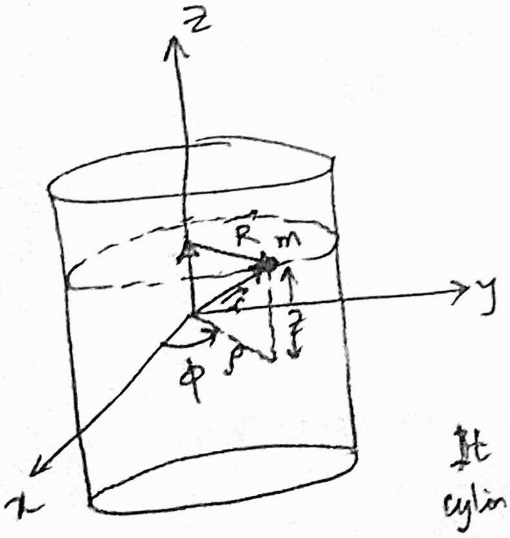
Hamilton's eq^{n}s of motion are:

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mgl\cos\theta \leftarrow$$

p_θ is not constant of motion here, since $\dot{p}_\theta$ explicitly depends on θ.

Example 4: Hooke's law force constrained to the surface of a cylinder:



Consider a mass m is attracted by a force directed towards the origin of a cylinder and proportional to the distance from the origin, and is constrained to move on the surface of the cylinder defined by

$$x^2 + y^2 = R^2$$

It is natural to transform this problem into cylindrical coordinates ρ, z, ϕ.

The force is just Hooke's law

$$\vec{F} = -k\vec{r}$$

The potential energy is the same as for the Harmonic oscillator,

$$U = \frac{1}{2}kr^2 = \frac{1}{2}k(\rho^2+z^2)$$

↳ this is independent of ϕ, thus ϕ is cyclic.

In cylindrical coordinates the velocity is

$$v^2 = \dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2$$

As the mass is confined to the surface of the cylinder, we have the constraint eq^n as

$$\rho = R$$

$$\Rightarrow \dot{\rho} = 0$$

Thus, the Lagrangian is

$$L = T - U = \frac{1}{2}m(R^2\dot{\phi}^2 + \dot{z}^2) - \frac{1}{2}k(R^2+z^2)$$

The generalized coordinates are ϕ and z (as ρ is replaced by the eq^n of constraint.)

The corresponding generalized momenta are:

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mR^2\dot{\phi}$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

The $$H = \sum_j p_j \dot{q}_j - L = \frac{p_\phi^2}{2mR^2} + \frac{p_z^2}{2m} + \frac{1}{2}k(R^2+z^2) = T+U=E.$$

This is expected as the system is conservative and the transformation from rectangular to cylindrical coordinates doesn't depend explicitly on time.

The Hamilton's eq^n's of motion are:

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \qquad \dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mR^2}$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -kz \qquad \dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}$$

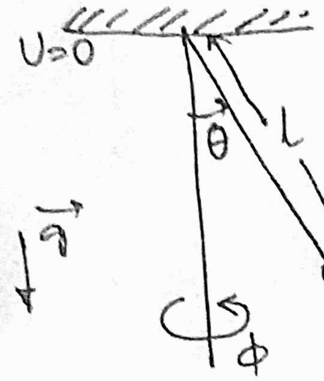
As $\dot{p}_\phi = 0 \Rightarrow p_\phi = mR^2\dot{\phi} = \text{Constant}$

thus the angular momentum about the axis of the cylinder is conserved — ϕ is a cyclic variable.

$$\dot{p}_z = -kz \Rightarrow m\ddot{z} = -kz \Rightarrow \boxed{\ddot{z} + \frac{k}{m}z = 0}$$

eq^n of SHM with angular frequency $\omega = \sqrt{k/m}$.

Example 5: Spherical pendulum with Hamiltonian Mechanics:



mass m & length l.

The generalized coordinates are θ, φ, since the length is fixed (eq^n of constraint) at r = l.

$$T = \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{1}{2} m l^2 \sin^2\theta \, \dot{\phi}^2$$

$$U = -mgl\cos\theta$$

Lagrangian, $L(r, \theta, \phi, \dot{r}, \dot{\theta}, \dot{\phi}) = \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{1}{2} m l^2 \sin^2\theta \, \dot{\phi}^2 + mgl\cos\theta$

Generalized momenta are:

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta} \qquad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m l^2 \sin^2\theta \, \dot{\phi}$$

Hamiltonian is

$$H(r, \theta, \phi, p_r, p_\theta, p_\phi) = \frac{p_\theta^2}{2ml^2} + \frac{p_\phi^2}{2ml^2 \sin^2\theta} - mgl\cos\theta$$

The eq^n's of motion are:

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2 \cos\theta}{2ml^2 \sin^3\theta} - mgl\sin\theta \quad \text{—— (i)}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \quad \text{—— (ii)}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2} \longrightarrow p_\theta = ml^2\dot{\theta} \Rightarrow \dot{p}_\theta = ml^2\ddot{\theta} \quad \text{—— (iii)}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{ml^2 \sin^2\theta} \longrightarrow p_\phi = ml^2 \sin^2\theta \, \dot{\phi} \quad \text{—— (iv)}$$

From eq^n's (i) & (iii),

$$ml^2\ddot{\theta} = \frac{p_\phi^2 \cos\theta}{2ml^2 \sin^3\theta} - mgl\sin\theta.$$

$$\Rightarrow \boxed{\ddot{\theta} - \frac{p_\phi^2 \cos\theta}{2m^2 l^4 \sin^3\theta} + \frac{g}{l}\sin\theta = 0.}$$

From eq's (ii) & (iv),

$$\dot{p}_\phi = \frac{d}{dt}\left(ml^2 \sin^2\theta \, \dot{\phi}\right) = 0$$

$$\Rightarrow \boxed{p_\phi = ml^2 \sin^2\theta \, \dot{\phi}^2 = \text{Constant.}}$$

That is the angular momentum about the vertical axis is conserved.