VARIATIONAL CALCULUS IN PHYSICS

- In ordinary calculus, we often work with real "functions", which are rules for mapping real numbers to real numbers;

$$\boxed{f : \mathbb{R} \to \mathbb{R}}$$

For example, the function $\sin x$ maps the whole of the real line to the interval $[-1, 1]$. Functions can have various properties — for instance, they can be continuous and differentiable, they can have stationary points, local maxima and minima.

- Functionals: In the calculus of variation, we work with "functionals", these are objects that map functions to the real line (like they "eat" the whole function $y(x)$ and return a single number). A functional J is a map :

$$\boxed{J : C^{\infty}(\mathbb{R}) \to \mathbb{R}}$$

where $C^{\infty}(\mathbb{R})$ is the space of smooth (having derivatives of all orders) functions. For example,

$$J[y] = \int_0^1 \left(y'(x)\right)^2 dx$$ — is a functional, for which any sufficiently well-behaved real function $y(x)$ gives a real number $J[y]$.

[Here and throughout the text, $y'(x)$ denotes the derivative $\frac{dy}{dx}$ evaluated at x]

The square bracket notation $J[y]$ is used to denote the functional J which depends on the function $y(x)$.

Functionals share many of the same properties as functions. In particular, the notion of a stationary point of a function has an important analogue in the theory of functionals, which gives rise to the calculus of variations, as we'll see later.

To find the function $y(x)$ that maximizes or minimizes a given functional $J[y]$, we need to define its "functional derivatives".

- **The functional derivative:**

We restrict ourselves to expressions of the form

$$J[y] = \int_{x_1}^{x_2} f(x, y, y', y'', ---, y^{(n)})\, dx$$

where f depends on the value of $y(x)$ and only finitely many of its derivatives. Such functionals are said to be "local" in x.

Consider first a functional $J = \int f(x, y, y')\, dx$ in which f depends only on x, y and y'. Make a change $y(x) \rightarrow y(x) + \epsilon\eta(x)$, where ϵ is a small x-independent constant. The resultant change in J is:

$$J[y+\epsilon\eta] - J[y]$$

$$= \int_{x_1}^{x_2} \{ f(x, y + \epsilon\eta, y' + \epsilon\eta') - f(x, y, y') \} dx$$

(Taylor series expansion of function with two variables)

$$= \int_{x_1}^{x_2} \left\{ \epsilon\eta \frac{\partial f}{\partial y} + \epsilon \underline{\frac{d\eta}{dx} \frac{\partial f}{\partial y'}} + O(\epsilon^2) \right\} dx$$

(integrating by parts)

$$= \int_{x_1}^{x_2} \epsilon\eta \frac{\partial f}{\partial y}\, dx + \left[\epsilon\eta \frac{\partial f}{\partial y'} \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \epsilon\eta \cdot \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx + O(\epsilon^2)$$

In order to simplify the result we'll assume, for the moment, that the end-points are "fixed", i.e. not only x_1, x_2 are given but also $y(x_1)$ and $y(x_2)$ are fixed. This restriction means that we require $\boxed{\eta(x_1) = \eta(x_2) = 0}$ ← (fixed end points assumption)

So, $$\left[\epsilon\eta \frac{\partial f}{\partial y'} \right]_{x_2}^{x_1} = 0$$

So,

$$J[y+\epsilon\eta] - J[y] = \int_{x_1}^{x_2} \epsilon\,\eta(x) \left\{ \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right\} dx + O(\epsilon^2)$$

Now define δJ to be the $O(\epsilon)$ part of $J[y+\epsilon\eta]-J[y]$ we have

$$\delta J = \int_{x_1}^{x_2} \underline{\epsilon\,\eta(x)} \left\{ \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right\} dx$$

$$= \int_{x_1}^{x_2} \underline{\delta y(x)} \cdot \left(\frac{\delta J}{\delta y(x)} \right) dx$$

where $\delta y(x) \equiv \epsilon\,\eta(x)$ [as $y(x) \to y(x) + \epsilon\eta(x)$]

The function

$$\boxed{\frac{\delta J}{\delta y(x)} = \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)}$$

is called the "functional derivative" of J with respect to the function $y(x)$.

- **The Euler – Lagrange Equation:**

Suppose we have a differentiable function $F(y_1, y_2, \ldots, y_n)$ of n variables and we want to find its "stationary points" — those being the locations at which F has its maxima, minima and saddlepoints. At the stationary point, the variation

$$\delta F = \sum_{i=1}^{n} \frac{\partial F}{\partial y_i}\,\delta y_i$$ must be zero, for all

possible δy_i. The necessary and sufficient condition for this to happen is that all partial derivatives $\frac{\partial F}{\partial y_i}$, $i = 1, \ldots, n$ be zero.

By analogy, we expect that a functional $J[y]$ will be "stationary" under fixed-endpoint variations $y(x) \longrightarrow y(x) + \delta y(x)$, when the functional derivative $\dfrac{\delta J}{\delta y(x)}$ vanishes for all x.

In other words,

$$\boxed{\frac{\partial f}{\partial y(x)} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'(x)}\right) = 0, \quad x_1 < x < x_2}$$

This condition for $y(x)$ to be a stationary point is usually called the "Euler-Lagrange" equation.

⇒ When the functional depends on more than one function y, then we'll need one Euler-Lagrange equation for each function $y_i(x)$:

$$\frac{\delta J}{\delta y_i(x)} = \frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) = 0$$

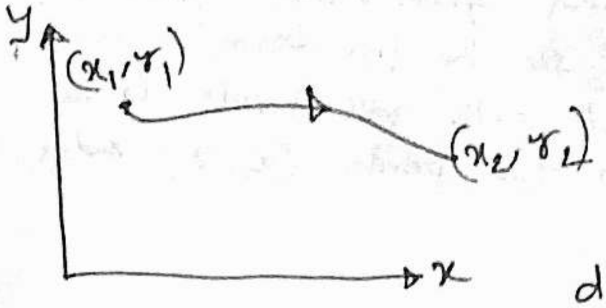
Say $i = 1, 2, 3, \ldots, N$. Thus each of the N eq^ns can be solved independently when N variables are independent.

⇒ If the function f depends on higher derivatives, y'', $y^{(3)}$, etc, then we have to integrate by parts more times, and we end up with

$$\frac{\delta J}{\delta y(x)} = \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) - \frac{d^3}{dx^3}\left(\frac{\partial f}{\partial y^{(3)}}\right) + \cdots = 0$$

Applications of Euler's Equation:

1) Example: Shortest distance between two points:



Consider the path lies in the x-y plane. The infinitesimal length of arc is

$$ds = \sqrt{dx^2 + dy^2}$$
$$= \left[\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,\right] dx$$

Then the length of the arc is

$$J = \int_1^2 ds = \int_1^2 \left[\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,\right] dx$$

The function f is

$$f = \sqrt{1+y'^2}$$

So, $\frac{\partial f}{\partial y} = 0$ and $\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$

Using the Euler-Lagrange eq^n:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$\Rightarrow 0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

$$\Rightarrow \frac{y'}{\sqrt{1+y'^2}} = \text{constant} = C$$

So, $\frac{y'^2}{1+y'^2} = C^2$

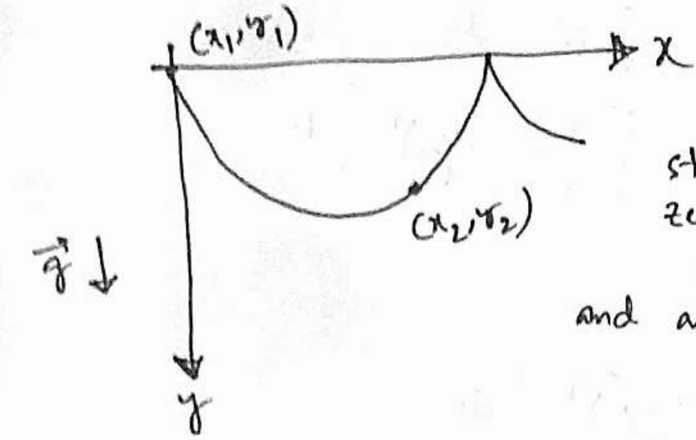
$$\Rightarrow C^2 + C^2 y'^2 = y'^2$$
$$\Rightarrow y'^2(1-C^2) = C^2 \Rightarrow y' = \frac{C}{\sqrt{1-C^2}} = a \text{ (say)}$$

Therefore $\boxed{y = ax + b}$

which is the eq^n of straight line — shortest path bet^n two points in a plane — this stationary value is a minimum.

(2) Example: Brachistochrone problem: This involves finding the path having the minimum transit time between two points. For simplicity, we take the case of frictionless motion in the x-y plane with ~~a~~ uniform gravitational field ~~along~~ acting in the $\hat{y}$ direction (see the fig. below). The question is what constrainted path will result in the minimum transit time between two points (x_1, y_1) and (x_2, y_2).

Bernoulli (1696)



Consider a particle of mass 'm' starts at origin $x_1 = 0, y_1 = 0$ with zero velocity.

Since the problem conserves energy and assuming that initially

$$E = KE + PE = 0$$

$$\Rightarrow \frac{1}{2}mv^2 - mgy = 0$$

$$\Rightarrow v = \sqrt{2gy}$$

So, the transit time will be

$$t = \int_{x_1}^{x_2} \frac{ds}{v} = \int_{x_1}^{x_2} \frac{\sqrt{dx^2+dy^2}}{\sqrt{2gy}}$$

functional (J)

$$= \int_{x_1}^{x_2} \sqrt{\frac{(1+x'^2)}{2gy}}\, dy, \quad \text{where } x' = \frac{dx}{dy}$$

Note that, here the independent variable is chosen to be y and the dependent variable is $x(y)$.

Change in potential energy for a force $mg\hat{y}$ is (work done by the force)

$$U_b - U_a = -\int_{y_a}^{y_b} (mg)\, dy$$

$$= -mg(y_b - y_a)$$

For $y_a = 0$, take. $U_a = 0$

~~and~~ $y_b = h$.

$$U(h) = -mgh.$$

The function f of the integral is

$$f \equiv f(y, x, x') = \frac{1}{\sqrt{2g}} \sqrt{\frac{1+x'^2}{y}}$$

So, $\frac{\partial f}{\partial x} = 0$, $\quad \frac{\partial f}{\partial x'} = \frac{1}{\sqrt{2g}} \cdot \frac{1}{\sqrt{y}} \cdot \frac{x'}{\sqrt{1+x'^2}}$

Therefore, the Euler-Lagrange eq^n will be

$$\frac{\partial f}{\partial x} - \frac{d}{dy}\left(\frac{\partial f}{\partial x'}\right) = 0$$

$$\Rightarrow \quad 0 - \frac{d}{dy}\cdot\left(\frac{x'}{\sqrt{y}.\sqrt{1+x'^2}}\right) = 0$$

So, $\dfrac{x'}{\sqrt{y}.\sqrt{1+x'^2}} = \text{Const.} = \dfrac{1}{\sqrt{2a}}$

$$\Rightarrow \quad \frac{x'^2}{y(1+x'^2)} = \frac{1}{2a}$$

$$\Rightarrow \quad 2ax'^2 - y - yx'^2 = 0$$

$$\Rightarrow \quad x'^2(2a-y) = y$$

$$\Rightarrow \quad x'^2 = \frac{y}{2a-y} \quad \Rightarrow \quad x' = \frac{y}{\sqrt{2ay-y^2}}$$

So, $x = \displaystyle\int_{y_1}^{y_2} \frac{y\,dy}{\sqrt{2ay-y^2}}$

change of variable $\quad y = a(1-\cos\theta)$

$$\Rightarrow dy = a\sin\theta\, d\theta$$

$$x = \int \frac{a^2(1-\cos\theta)\sin\theta}{a\sin\theta}\,d\theta$$

$$= \int a(1-\cos\theta)\,d\theta.$$

$2ay-y^2$
$= 2a^2(1-\cos\theta) - a^2(1+\cos^2\theta - 2\cos\theta)$
$\Rightarrow a^2 - a^2\cos^2\theta.$
$\Rightarrow a^2\sin^2\theta.$
$\sqrt{2ay-y^2} = a\sin\theta.$

$$\Rightarrow \quad x = a(\theta - \sin\theta) + \text{Constant}.$$

So, we get
$$\begin{cases} x = a(\theta - \sin\theta) + \text{Const.} \\ y = a(1-\cos\theta). \end{cases}$$
which are the parametric equations for a "cycloid"

Therefore, the shortest time between two points is obtained by constraining the motion to follow a cycloid shape. The mass first accelerates rapidly by falling down steeply, and then follows the curve and coasts upward at the end.

- Selection of the independent variable:

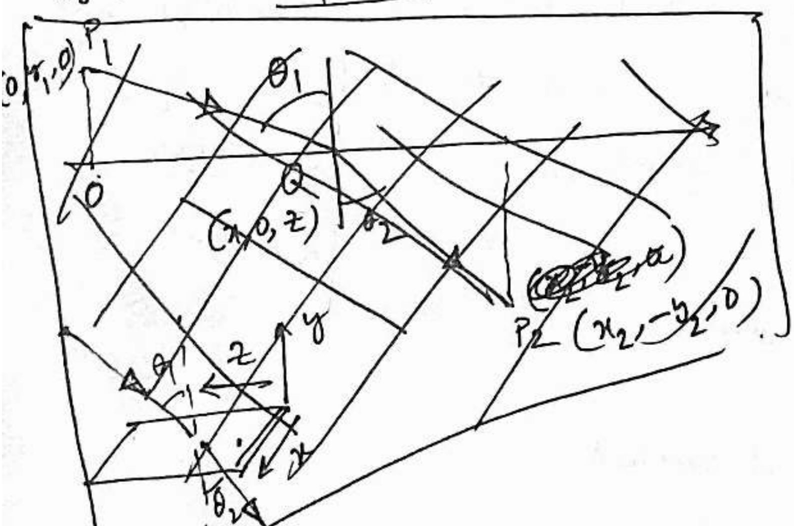
A wide selection of variables can be chosen as the independent variable for variational calculus. The derivation of Euler-Lagrange eq^n and example I, both assumed the independent variable as x, whereas example 2 used y as the independent variable. Lagrange mechanics uses time t as the independent variable. Selection of which variable to use as the independent variable doesn't change the physics of a problem, but some selection can simplify the mathematics for obtaining an analytic solution. ~~see next example for this~~.

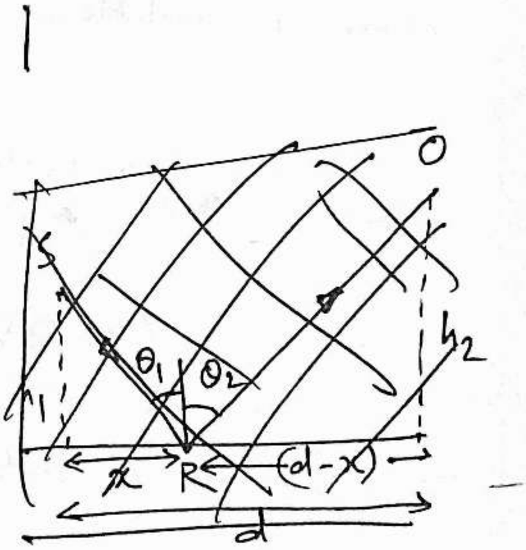
~~(3) Example: Surface area of a cylindrically-symmetric soap bubble:~~

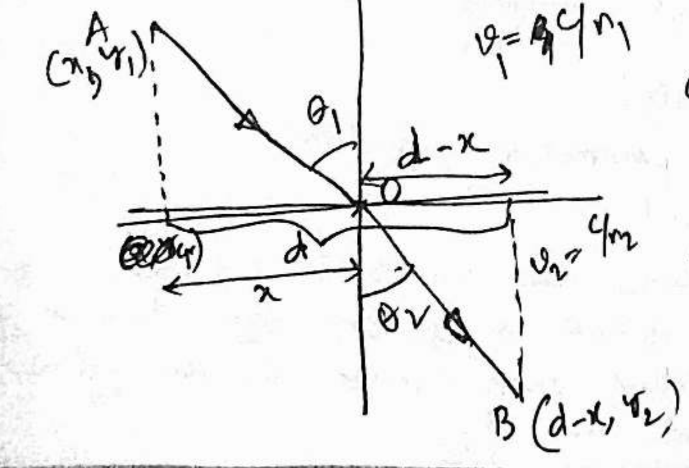
(3) Example: Fermat's Principle:

(~~functions with several independent variables~~)

In 1662 Fermat proposed that the propagation of light obeyed the generalized principle of least transit time, that is the path taken between two points by a ray of light is the path that can be traversed in the least time.







Consider the light travels from point $A(x, y_1)$ to the point $B(d-x, y_2)$. The light beam intersects a plane glass interface at the point O

Fermat discovered that the required path travelled by light is the path for which the travel time t is minimum.

The transit time t from initial point A to the final point B is given by

$$t = \int_A^B dt = \int_A^B \frac{ds}{v} = \frac{1}{c}\int_A^B n\,ds$$

→ depends on the media.

$\boxed{v = c/n}$, where n is the refractive index

1st Method

I

Using Pythagoras' theorem to the triangles, we can find out the length of the paths as

$$AO = \sqrt{x^2 + y_1^2}$$

and $$OB = \sqrt{(d-x)^2 + y_2^2}$$

The total transit time is given by

$$t = \frac{1}{c}\int_A^B n\,ds$$

$$= \frac{1}{c}\cdot n_1 \underbrace{\int_A^O ds}_{AO} + \frac{1}{c}\cdot n_2 \underbrace{\int_O^B ds}_{OB}$$

$$= \frac{1}{c}\left(n_1\,AO + n_2\,OB\right)$$

$$= \frac{1}{c}\left(n_1\sqrt{x^2+y_1^2} + n_2\sqrt{(d-x)^2+y_2^2}\right)$$

This problem involves one dependent variable $y(x)$. The find the minima, set the partial derivatives

$$\frac{\partial t}{\partial x} = 0$$

So, $$\frac{\partial t}{\partial x} = \frac{1}{c}\left(\frac{n_1 x}{\sqrt{x^2+y_1^2}} - \frac{n_2 (d-x)}{\sqrt{(d-x)^2+y_2^2}}\right) = 0$$

$$\Rightarrow \quad n_1 \sin\theta_1 - n_2 \sin\theta_2 = 0$$

So, $$\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2}$$

known as "Snell's law"

2nd Method II

Using Euler – Lagrange eq^n.

transit time

$$t = \int_A^B dt = \frac{1}{c}\int_A^B n\,ds$$

$$= \frac{1}{c}\int_A^B n(x,y)\sqrt{dx^2+dy^2}$$

$$= \frac{1}{c}\int_A^B n(x,y)\sqrt{1+(x')^2}\,dy \qquad \left(\text{where } x' = \frac{dx}{dy}\right)$$

(here we chose y as independent variable and $x(y)$ as dependent variable)

$$\Rightarrow t = \frac{1}{c}\left[\int_{y_1}^{0} n_1\sqrt{1+(x')^2}\,dy + \int_0^{y_2} n_2\sqrt{1+(x')^2}\,dy\right]$$

~~The function is~~ $\mathcal{L} = \mathcal{L}\frac{1}{c}$ here t is the functional, which ~~depends~~ is a function of x' only, not x. So, the Euler-Lagrange eq^n simplifies to

$$0 + \frac{d}{dy}\left(\frac{1}{c}\cdot\frac{n_1 x'}{\sqrt{1+x'^2}} + \frac{1}{c}\,\frac{n_2 x'}{\sqrt{1+x'^2}}\right) = 0$$

But $x' = \frac{dx}{dy} = \tan\theta_1$ for n_1 (see the figure)

and $x' = -\tan\theta_2$ for n_2.

So,
$$\frac{d}{dy}\cdot\frac{1}{c}\left(\frac{n_1\tan\theta_1}{\sqrt{1+\tan^2\theta_1}} - \frac{n_2\tan\theta_2}{\sqrt{1+\tan^2\theta_2}}\right) = 0$$

therefore, $\frac{1}{c}\cdot\left(n_1\sin\theta_1 - n_2\sin\theta_2\right) = \text{Constant}.$

this constant must be zero, as for $n_1 = n_2$, $\theta_1 = \theta_2$

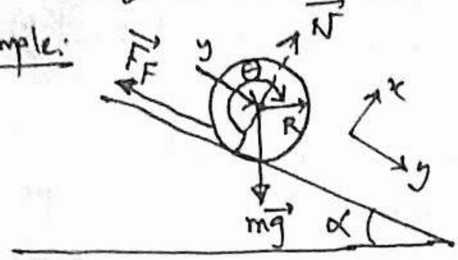
Thus, Fermat's principle leads to "Snell's law"

$$\boxed{n_1\sin\theta_1 = n_2\sin\theta_2}$$

Constrained variational systems:

Imposing a constraint on a variational system implies that N constrained coordinates $y_i(x)$ are correlated, i.e. they are not independent. Constrained motion implies that constraint forces must be acting to account for the correlation of the variables. These constraint forces must be taken into account in the equations of motion.

Example:



A disk is rolling down an inclined plane without slipping.

There are three coordinates:
x (perpendicular to the inclined plane)
y (along the surface of the inclined plane)

and the rotation angle θ. The constraint forces are $\vec{F}_F$ and $\vec{N}$ — these lead to the correlation of the variables such that $x = R$ and $y = R\theta$.

Basically there is only one idependent variable, which can be either y or θ.

The use of only one independent variable essentially omits the constraint forces, which is fine if we only need to know the equation of motion. If we need to determine the forces of constraint, then it is necessary to include all coordinates explicitly in the equations of motion.

① Holonomic constraints:

The constraints mean that the coordinate $y_i(x)$ are not independent, but are related by "equations of constraint". A constraint is called "holonomic" if the equations of constraint can be expressed in the form of an algebric equation that directly and ~~exact~~ unambiguously specifies the shape of the surface of constraint.

These types of constraints can be expressed in the form of algebric relations that directly specify the shape of the surface of constraint in coordinate space $q_1, q_2, \ldots q_j, \ldots q_n$.

$$g_k(q_1, q_2, \ldots, q_j, \ldots q_n; t) = 0$$

where $j = 1, 2, 3, \ldots, n$. There can be m such equations of constraint where $0 \leq k \leq m$.

Example of such holonomic constraint is if the motion is confined to the surface of a sphere of radius R which can be written in the form

$$g = x^2 + y^2 + z^2 - R^2 = 0$$

- Non-holonomic constraints:

Such constraints do not provide an algebric equation between the correlated coordinates, they usually comes with an "inequality".

For example, if a particle's location is restricted to lie inside a spherical shell of radius R, it can be expressed as

$$g = x^2 + y^2 + z^2 - R^2 \leq 0$$

- Time dependence:

A constraint is called "scleronomic" if the constraint is not explicitly time dependent. Fortunately, a major fraction of systems are scleronomic. On the other hand, the constraint is called "rheonomic" if it is explicitly time-dependent. An example of a rheonomic system is where the size or shape of the surface of constraint is explicitly time dependent, such as a deflating or inflating tire.

- Treatment of constraint forces in variational calculus:

Mainly 3 approaches:

(I) Generalized coordinate approach: It exploits the correlation of the n coordinates due to the m no. of constraint forces (holonomic constraint) to reduce the dimension of the equations of motion to $s = n - m$ degrees of freedom. This approach embeds the m constraint forces into the choice of generalized coordinates and doesn't determine the constraint forces.

(II) Lagrange multiplier approach: It exploits generalized coordinates but includes the m constraint forces into the Euler-Lagrange eq^{n}s to determine both the constraint forces in addition to the n no. of eq^n of motion.

(III) Generalized forces approach: It treats constraint forces as well as other forces explicitly.

Generalized coordinates in variational calculus:

→ Newtonian mechanics is based on a vectorial treatment of mechanics which is difficult to apply when solving complicated problems. Constraint forces acting on a system usually are unknown, and thus must be included explicitely in Newtonian mechanics so that they can be determined simultaneously with the solution of dynamical eq^n's of motion.

→ On the other hand, in variational approaches, the solution of the dynamical eq^n's of motion can be simplified by introducing n independent generalized coordinates. These generalized coordinates can be any set of <u>independent variables</u>, q_i, where $1 \le i \le n$, plus the corresponding velocities $\dot{q}_i$ (for Lagrangian mechanics) or generalized momenta p_i (for Hamiltonian mechanics).

The generalized coordinates used for variational approach <u>do not need to be orthogonal</u>, they only <u>need to be independent</u>. This is the main advantage of this approach over Newtonian mechanics. For example, generalized coordinates can be chosen to be perpendicular to a corresponding constaint force (see the figure of motion along the inclined plane in the last section, generalized coordinate x was chosen to be perpendicular of frictional force $\vec{F}_f$ and y to be perpendicular of normal force $\vec{N}$), and therefore that specific constraint force does no work for motion along that generalized coordinate.

Suppose, a system has 'm' no. of equations of constraints which are <u>holonomic</u>, then m algebraic constraint relations can be used to transform the coordinates into $s = n - m$ independent generalized coordinates $q_1(x), q_2(x), \ldots, q_s(x)$, where the actual problem has 'n' no. of spatial coordinates $y_1(x), y_2(x), \ldots, y_n(x)$. We also need the generalized first derivatives $\dot{q}'_1(x), \dot{q}'_2(x), \ldots, \dot{q}'_s(x)$ to formulate the Euler-Lagrange equations for each independent parameter q_i

$$\boxed{\frac{\partial f}{\partial q_i} - \frac{d}{dx}\frac{\partial f}{\partial q'_i} = 0}$$

where $i = 1, 2, 3, \ldots, s$ — there are $s = n - m$ such Euler-Lagrange eq^n.

Lagrange equations from Hamilton's Principle:

Hamilton's Principle (1834, 1835) states that dynamical systems follow paths that minimize the time integral of the Lagrangian. That is, the action functional S

$$\boxed{S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt}$$

$q \equiv$ generalized coordinates, $\dot{q} \equiv$ their velocities, $t \equiv$ time.

has a minimum value for the correct path of motion. For our purpose, we'll use the standard Lagrangian function as

$$\boxed{L \equiv T - U}$$

From Hamilton's principle,

$$\boxed{\delta S = \delta \int_{t_1}^{t_2} L\, dt = 0}$$

Therefore, variational calculus implies that a system of 's' independent generalized coordinates must satisfy the basic Euler-Lagrange eq^n,

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0} \qquad j = 1, 2, \cdots, s$$

Note that, the above eq^n can be rigorously derived from "d'Alembert's principle of virtual work" with help of transformation of generalized coordinates (refer to any standard textbooks).

Full treatment:

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = \underbrace{\sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(q, t)}_{\text{Lagrange multiplier}} + \underbrace{Q_j^{EXC}}_{\text{Generalized force}}}$$

for n variables with m equations of constraint:

$$g_k(q_1, q_2, \cdots, q_n, t) = 0$$

$$Q_j^{EXC} = \sum_i^n \vec{F}_i^A \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

- One can omit the Q_j^{EXC} term if the system is conservative
- ~~One can omit Lagrange multiplier term~~ and for holonomic eq^n of constraint.

Applications to unconstrained systems:

Although most dynamical systems involve constrained motion, it's useful to consider examples of systems subject to conservative forces with no constraints. For this case, the generalized coordinates are chosen same as the normal coordinates.

Example 1: Motion of a free particle, $U = 0$:

Lagrangian in Cartesian coordinates.

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial L}{\partial \dot{y}} = m\dot{y}, \quad \frac{\partial L}{\partial \dot{z}} = m\dot{z} \quad \Bigg| \quad \frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial z} = 0$$

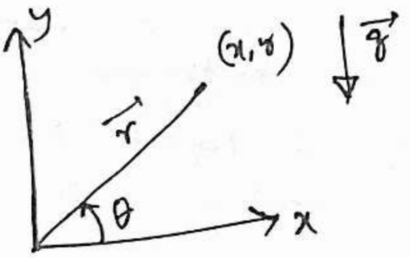
So,
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \frac{d}{dt}(m\dot{x}) - 0 = 0$$

Similary for y & z ~~generalized~~ coordinate.

We set,
$$p_x = m\dot{x} = \text{constant}$$
$$p_y = m\dot{y} = \text{constant}$$
$$p_z = m\dot{z} = \text{constant}$$

This shows that the linear momentum is conserved if U is constant, that is $\vec{F}^a = 0$. Note that, momentum conservation has been derived without any direct reference to forces.

Example 2: Motion in a uniform gravitational field:



Consider the motion is in the x-y plane

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$U = mgy \quad \text{with} \quad U(y=0) = 0$$

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy$$

Euler-Lagrange eq^n for x-coordinate:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \frac{d}{dt} m\dot{x} - 0 = 0$$

Thus the horizontal momentum $p_x = m\dot{x}$ is conserved.

Euler-Lagrange eq^n for y-coordinate:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = \frac{d}{dt} m\dot{y} + mg = 0$$

$$\Rightarrow \boxed{\ddot{y} = -g} \longleftarrow \text{same as derived using Newton's law of motion.}$$

(*) What would happen if we choose the polar coordinates r, θ for this problem ?

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(r\dot{\theta})^2$$

$$U = mgr\sin\theta$$

$$L = T - U = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mgr\sin\theta$$

Euler-Lagrange eq^n for r coordinate,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$$

$$\Rightarrow m\ddot{r} - mr\dot{\theta}^2 + mg\sin\theta = 0$$

$$\Rightarrow \ddot{r} - r\dot{\theta}^2 + g\sin\theta = 0 .$$

and for θ coordinate,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\cancel{\Rightarrow \frac{d}{dt}(mr^2\ddot{\theta})} \Rightarrow \frac{d}{dt}(mr^2\dot{\theta}) + mgr\cos\theta = 0$$

$$\Rightarrow 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} + mgr\cos\theta = 0$$

$$\Rightarrow 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} + gr\cos\theta = 0$$

These eq^n's written in polar coordinates are more complicated than the same expressed in Cartesian coordinates. This is because the potential energy U depends directly on the y coordinate, whereas it is a function of both r, θ.

—— This example illustrates the importance of choosing a ~~right set~~ suitable set of generalized coordinates. (Same thing what we discussed earlier for selection of independent variables in variational calculus.).

Example 3 : Central forces :

Consider a mass 'm' moving under the influence of a spherically-symmetric, conservative, attractive, inverse-square force. Potential energy

$$U = -\frac{k}{r}$$

It's natural to express the Lagrangian in spherical coordinates (r, θ, ϕ) for this system.

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(r\dot{\theta})^2 + \frac{1}{2}m(r\sin\theta\,\dot{\phi})^2 + \frac{k}{r}$$

Euler-Lagrange eq^n for r-coordinate gives,

$$m\ddot{r} - mr\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) = \frac{k}{r^2}$$

where the $mr\sin^2\theta\,\dot{\phi}^2$ term comes from the "centripetal acceleration".

E-L eq^n for ϕ-coordinate gives,

$$\frac{d}{dt}\left(mr^2\sin^2\theta\,\dot{\phi}\right) = 0$$

this implies the angular momentum about ϕ axis, i.e.

$$p_\phi = mr^2\sin^2\theta\,\dot{\phi} = \text{constant}$$ — is a constant of motion.

E-L eq^n for θ-coordinate gives,

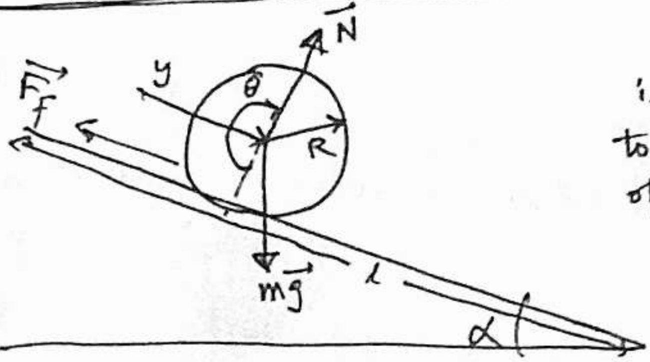
$$\frac{d}{dt}\underline{(mr^2\dot{\theta})} - mr^2\sin\theta\cos\theta\,\dot{\phi}^2 = 0$$

$$\Rightarrow \dot{p}_\theta = mr^2\sin\theta\cos\theta\,\dot{\phi}^2 = \frac{p_\phi^2\cos\theta}{2mr^2\sin^3\theta}.$$

Note that, p_θ is a constant of motion if $p_\phi = 0$.

Applications to systems involving holonomic constraints:

Example 4: Disc rolling on an inclined plane:



Disc rolling without sliding — i.e. assuming the friction is sufficient to ensure that the rolling equation of constraint applies.

Disc has radius R and moment of inertia $I = \frac{1}{2} mR^2$.

Generalized coordinates are: y (distance along the inclined plane, which is perpendicular to the normal force $\vec{N}$), x (perpendicular to the inclined plane) and θ (rolling angle).

The constraint eq^n for rolling is holonomic:

$$y - R\theta = 0$$

The frictional force is $\vec{F_F}$. The constraint that it rolls along the plane implies

$$x - R = 0 \leftarrow \text{holonomic.}$$

As there are 3 generalized coordinates and 2 eq^n of constraint, the system will have only one generalized coordinate

(a) Newton's laws of motion:

Newton's law for the components of the forces along the inclined plane (y-dir^n):

$$mg \sin\alpha - F_F = m\ddot{y} \quad \text{—— (i)}$$

Newton's law for the components of the forces perpendicular to the inclined plane (x-dir^n):

$$mg \cos\alpha = N \quad (\text{since there is no motion along } x\text{-dir}^n)$$

The torque on the disc gives

$$F_F R = I\ddot{\theta}$$

Assuming the disc ~~rolls~~ rolls (i.e eq^n of constraint) gives

$$\ddot{y} = R\ddot{\theta}$$

Then, $$F_F = \frac{I}{R^2}\ddot{y}$$

Inserting this into eq^n (1) gives

$$\left(m + \frac{I}{R^2}\right)\ddot{y} - mg\sin\alpha = 0$$

The moment of inertia of a uniform solid circular disc is

$$I = \frac{1}{2}mR^2$$

So, $$\left(m + \frac{1}{2}m\right)\ddot{y} - mg\sin\alpha = 0$$

$$\Rightarrow \quad \ddot{y}\cdot\frac{3}{2}m = mg\sin\alpha$$

$$\Rightarrow \boxed{\ddot{y} = \frac{2}{3}g\sin\alpha}$$

and the frictional force is

$$\boxed{F_F = \frac{mg\sin\alpha}{3}} \leftarrow$$ Smaller than the gravitional force along the plane which is $mg\sin\alpha$ — that's why it rolls.

Both the eq^n of motion and forces have been determined.

(b) Lagrange eq^{n}s of motion with generalized coordinates:

The total kinetic energy

$$T = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2$$

Potential energy

$$V = mg(l-y)\sin\alpha$$

Thus the Lagrangian is

$$L = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2 - mg(l-y)\sin\alpha.$$

The holonomic eq^{n}s of constraint are

$$g_1 = y - R\theta = 0 \longrightarrow \dot{\theta} = \dot{y}/R$$

$$g_2 = x - R = 0$$

Holonomic constraints can be used to reduce this system to a single generalized coordinate y plus generalized velocity $\dot{y}$.
Expressing the Lagrangian in terms of this single generalized coordinate gives,

$$L = \frac{1}{2}\left(m + \frac{I}{R^2}\right)\dot{y}^2 - mg(l-y)\sin\alpha$$

Then the E-L eq^n gives,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0$$

$$\Rightarrow \left(m + \frac{I}{R^2}\right)\ddot{y} - mg\sin\alpha = 0$$

again $I = \frac{1}{2}mR^2$,

So, $\left(m + \frac{1}{2}m\right)\ddot{y} = mg\sin\alpha$

$$\Rightarrow \boxed{\ddot{y} = \frac{2}{3}g\sin\alpha}$$

Same as what we got using Newton's laws of motion,

Note that, no forces have been determined using this method.

(c) Lagrange eq^n with Lagrange multipliers:

In this approach the holonomic constraints are taken into account using Lagrange multipliers, instead of reducing the number of ~~independent~~ generalized coordinates.

~~Ignoring~~ Ignoring the trivial x-dependence (which is $x=R$), the Lagrangian is (same as before):

$$L = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2 - mg(l-y)\sin\alpha$$

The constraint eq^n's are:

$$g_1 = y - R\theta = 0 \longrightarrow \frac{\partial g_1}{\partial y} = 1 \quad \Big| \quad \frac{\partial g_1}{\partial \theta} = -R$$

$$g_2 = x - R = 0. \longrightarrow \frac{\partial g_2}{\partial y} = 0 \quad \Big| \quad \frac{\partial g_2}{\partial \theta} = 0$$

Here we'll take both y and θ as generalized coordinates.

E-L eq^n for y-coordinate using Lagrange multiplier,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = \lambda_1 \frac{\partial g_1}{\partial y} + \lambda_2 \cdot 0$$

$$\Rightarrow m\ddot{y} - mg\sin\alpha = \lambda_1 \quad \text{—(i)}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}$$

$m \Rightarrow$ no. of eq^n's of constraint.

E-L eqⁿ for θ-coordinate using Lagrange multiplier,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \lambda_1 \frac{\partial g_1}{\partial \theta} + \lambda_2 \cdot 0$$

$$\Rightarrow I\ddot{\theta} = -\lambda_1 R \quad \text{——— (ii)}$$

The constraint eqⁿ gives : $\ddot{y} = R\ddot{\theta}$ and $I = \frac{1}{2}mR^2$

So, eqⁿ (ii) becomes.

$$-\lambda_1 R = \frac{1}{2}mR^2 \cdot \ddot{y} \cdot \frac{1}{R}$$

$$\Rightarrow \lambda_1 = -\frac{1}{2}m\ddot{y}$$

$$\Rightarrow m\ddot{y} = -2\lambda_1 \quad \text{——— (iii)}$$

putting eqⁿ (iii) into eqⁿ (i) gives,

$$-2\lambda_1 - mg\sin\alpha = \lambda_1$$

$$\Rightarrow 3\lambda_1 = -mg\sin\alpha$$

$$\Rightarrow \lambda_1 = -\frac{mg}{3}\sin\alpha$$

The frinctional force is given by,

$$\boxed{F_f = \lambda_1 \frac{\partial g_1}{\partial y}} = \lambda_1$$

$$\Rightarrow \boxed{F_f = -\frac{mg}{3}\sin\alpha}$$

From eqⁿ (i),

$$m\ddot{y} = \lambda_1 + mg\sin\alpha$$

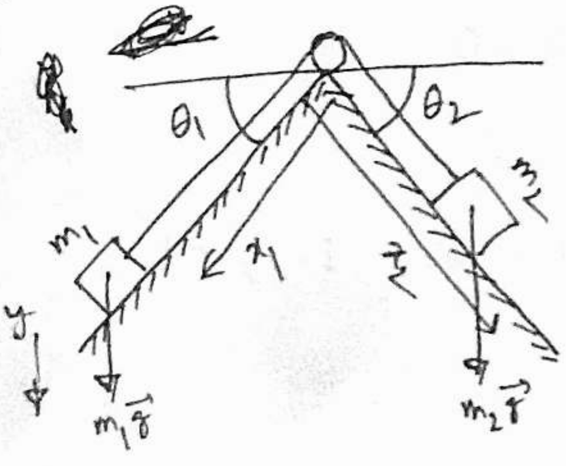
$$= -\frac{mg}{3}\sin\alpha + mg\sin\alpha$$

$$\Rightarrow m\ddot{y} = \frac{2}{3}mg\sin\alpha\,.$$

$$\Rightarrow \boxed{\ddot{y} = \frac{2}{3}g\sin\alpha}$$

→ Both the eqⁿs of motion and forces have been determined using this approach.

Example 5: Two connected masses on frictionless inclined planes:



In this problem, the generalized coordinates are: x_1 and x_2, which are perpendicular to the normal constraint forces on the inclined plane.

The holonomic constraint comes from the assumption that the length of the rope connecting the masses is constant.

So, the eqn of constraint is:

$$x_1 + x_2 - l = 0$$

(Note that, there are other constraints in this problem, they ensure that the two masses slide directly down the inclined planes, but we are not concerned about those here as those are responsible for motion along y-dirn which isn't important here.)

So, we can use only one variable x_1 or x_2.

Let us chose x_1 as the generalized coordinate.

(θ_1, θ_2 are given) constants.

$$x_2 = l - x_1$$

$$y_1 = x_1 \sin\theta_1$$

$$y_2 = (l - x_1)\sin\theta_2$$

The potential energy is given by

$$U = -m_1 g y_1 - m_2 g y_2$$

$$= -m_1 g x_1 \sin\theta_1 - m_2 g (l - x_1)\sin\theta_2$$

as, $x_2 = l - x_1 \Rightarrow \dot{x}_1 = -\dot{x}_2$

So, the total kinetic energy is given by,

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$= \frac{1}{2}(m_1 + m_2)\dot{x}_1^2$$

The Lagrangian is :

$$L = T - U$$

$$= \frac{1}{2}(m_1+m_2)\dot{x}_1^2 + m_1 g x_1 \sin\theta_1 + m_2 g (l - x_1)\sin\theta_2$$

$$\frac{\partial L}{\partial \dot{x}_1} = (m_1+m_2)\dot{x}_1$$

$$\frac{\partial L}{\partial x_1} = m_1 g \sin\theta_1 - m_2 g \sin\theta_2$$

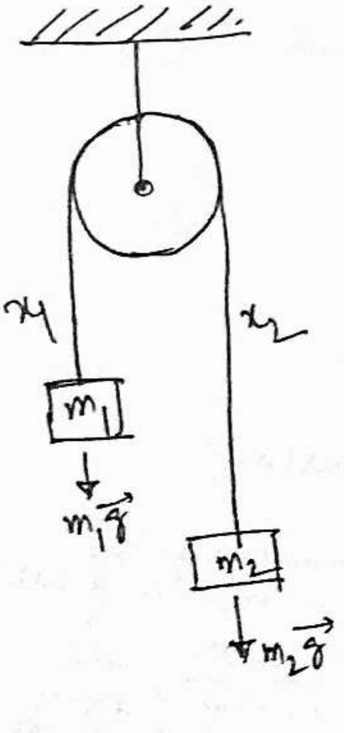
Thus E-L eq^n becomes.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = 0$$

$$\Rightarrow \quad (m_1+m_2)\ddot{x}_1 - g(m_1\sin\theta_1 - m_2\sin\theta_2)$$

$$\Rightarrow \boxed{\ddot{x}_1 = \frac{g}{m_1+m_2}\cdot(m_1\sin\theta_1 - m_2\sin\theta_2)} \quad \text{--- (i)}$$

(*) "special case" of this problem is the "Atwood's machine" with a massless pulley shown in the adjacent figure.



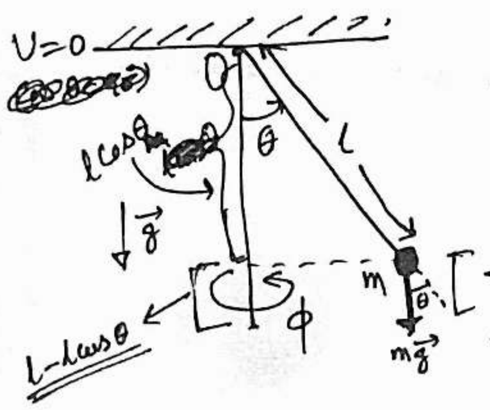
For this case, $\theta_1 = \theta_2 = 90°$

Thus eq^n of motion becomes (from eq^n (i)),

$$\boxed{\ddot{x}_1 = \frac{m_1 - m_2}{m_1+m_2}\, g}$$

Note that, these problems have been solved without any reference to the force in the rope (i.e. tension) or the "normal" constraint forces on the inclined planes.

Example 6 : Spherical pendulum :



The spherical pendulum is a classic holonomic problem in mechanics that involves rotation + oscillation, where the pendulum is free to swing in any direction.

[This also applies to a particle constrained to slide in a smooth frictionless spherical bowl under gravity]

Mass of the spherical pendulum is m and its length is l. The most convenient generalized coordinates are r, θ, ϕ with origin at point O (see the fig.).

Holonomic constraint eq^n is $r = l$, so the generalized coordinate is reduced to be θ and ϕ.

Kinetic energy, $T = \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{1}{2} m l^2 \sin^2\theta \, \dot{\phi}^2$

The potential energy $U = -mgl\cos\theta$.

Actually $U = mgl - mgl\cos\theta$
$= U_0 - mgl\cos\theta$.
choose $U_0 = 0$

Lagrangian is given as

$$L = T - U = \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{1}{2} m l^2 \sin^2\theta \, \dot{\phi}^2 + mgl\cos\theta$$

E-L eq^n for θ-coordinate,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow \quad m l^2 \ddot{\theta} = m l^2 \sin\theta\cos\theta \, \dot{\phi}^2 - mgl\sin\theta$$

E-L eq^n for ϕ-coordinate,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0$$

$$\Rightarrow \frac{d}{dt}\left(m l^2 \sin^2\theta \, \dot{\phi}\right) = 0$$

which gives $\quad m l^2 \sin^2\theta \, \dot{\phi} = p_\phi = \text{constant}.$

angular momentum for the pendulum rotating in ϕ-dir^n.

Thus the angular momentum p_ϕ is a conserved quantity, which is expected from Newton's law since there are no external torques applied.

The eq^n of motion for θ can be simplified to

$$ml^2\ddot{\theta} + mgl\sin\theta - ml^2 \sin\theta\cos\theta\,\dot{\phi}^2 = 0$$

$$\Rightarrow ml^2\ddot{\theta} + mgl\sin\theta - ml^2\sin\theta\cos\theta \cdot \frac{p_\phi^2}{m^2 l^4 \sin^4\theta} = 0$$

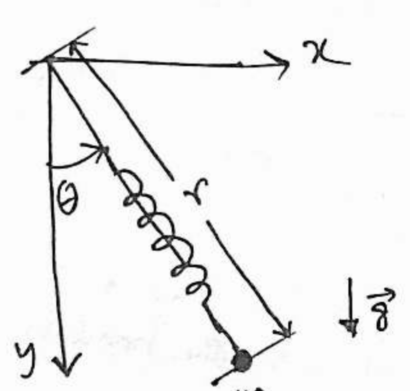
$$\left(\text{as } p_\phi = ml^2\sin^2\theta\,\dot{\phi}\right)$$

$$\Rightarrow \boxed{\ddot{\theta} + \frac{g}{l}\sin\theta - \frac{p_\phi^2\cos\theta}{m^2 l^4 \sin^3\theta} = 0}$$

(i) If $p_\phi = 0$, then the eq^n reduces to the simple harmonic pendulum, i.e. $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$

(ii) When $\ddot{\theta} = 0$, the motion is that of a "conical pendulum" which rotates at a constant angle, say θ_0, to the ~~verticla~~ vertical axis.

Example 7: Spring plane pendulum;



A mass m is suspended by a spring with spring constant k in the gravitational field. Besides the longitudinal spring vibration, the spring performs a plane pendulum motion in the vertical plane.

The system is holonomic, conservative (and scleronomic).

Introduce plane polar coordinates with radial length r and polar angle θ as generalized coordinates.

$$y = r\cos\theta$$
$$x = r\sin\theta$$

generalized coordinates r, θ are related to the Cartesian coordinates by these eq^{n}s.

The kinetic energy

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$$

The potential energy will be due to both the gravitational and spring motion.

$$U = -mgr\cos\theta + \frac{1}{2} k (r - r_0)^2$$

where r_0 denotes the rest length of the spring, $k \equiv$ spring constant.

Thus the Lagrangian is

$$L = T - U = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + mgr\cos\theta - \frac{1}{2} k (r - r_0)^2$$

For θ-coordinate, E-L eq^n gives:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt} (m r^2 \dot{\theta}) = -mgr\sin\theta$$

$$\Rightarrow \dot{p}_\theta = -mgr\sin\theta.$$

where $p_\theta = m r^2 \dot{\theta}$ is the angular momentum.

$$m r^2 \ddot{\theta} + 2 m r \dot{r} \dot{\theta} = -mgr\sin\theta$$

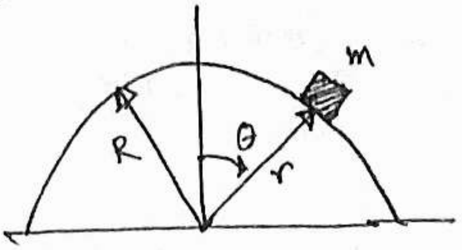
$$\Rightarrow m r^2 \ddot{\theta} = -mgr\sin\theta \underbrace{- 2 m r \dot{r} \dot{\theta}}_{\text{"Coriolis force" caused by the variation of the pendulum length.}}$$

For r-coordinate, E-L eq^n gives:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$$

$$\Rightarrow \underbrace{m\ddot{r}}_{\text{Tension in the spring, } F = m\ddot{r}} = \underbrace{m r \dot{\theta}^2}_{\text{centrifugal radial acceleration}} + \underbrace{mg\cos\theta}_{\text{Component of the gravitational force}} \underbrace{- k (r - r_0)}_{\text{represents the Hooke's law for the spring. } (F \propto -(r - r_0))}$$

Example 8 : Mass sliding on a frictionless spherical shell :



Consider a mass 'm' starts from rest at the top of a frictionless ~~shaper~~ spherical shell of radius R.

In this problem, the constraint is holonomic in a restricted domain, i.e as long as the force of constraint holds the mass against the surface of the spherical shell. ~~In gener~~

In general, non-holonomic constraints can be handled by using the generalized forces Q_j^{EXC} approach in E-L eq^n. However, we shall restrict ourselves to the domain where the system is holonomic.

r, θ are the obvious choice for generalized coordinates.

Lagrangian is

$$L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) - mgr\cos\theta$$

This Lagrangian is same irrespective of whether the constraint is holonomic or not.

The holonomic constraint is

$$g(r, \theta) = r - R = 0$$

So, we can reduce the generalized coordinates to just one $\rightarrow \theta$.

$$L = \frac{1}{2} m R^2 \dot{\theta}^2 - mgR\cos\theta$$

E-L eq^n for θ-coordinate,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow mR^2 \ddot{\theta} = mgR\sin\theta$$

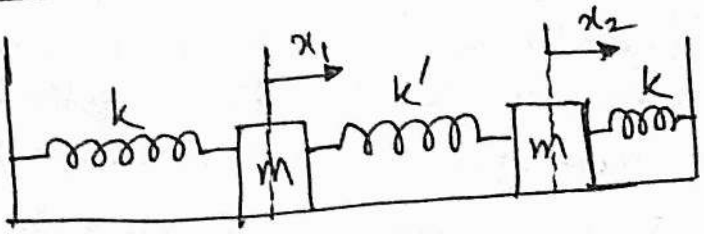
$$\Rightarrow \ddot{\theta} = \frac{g}{R}\sin\theta$$

Now $\ddot{\theta} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\dot{\theta}}{d\theta}$

$$\int \ddot{\theta}\, d\theta = \int \dot{\theta}\, d\dot{\theta} = \frac{g}{R}\int \sin\theta\, d\theta \Rightarrow \boxed{\dot{\theta}^2 = \frac{2g}{R}(1-\cos\theta)}$$

assuming $\dot{\theta} = 0$ at $\theta = 0$

Example 9: Linear systems of masses and springs:



← Coupled Linear Harmonic Oscillator.

Two blocks, each of mass 'm', sitting on a frictionless horizontal surface, are attached to three springs with spring constants for outer springs are k and spring constant for inner spring being k'. When the blocks are at rest, the spring are unstreched.

Let the displacements of block 1 and block 2 from their equilibrium positions be x_1 and x_2, respectively, each positive to the right. Generalized coordinates are x_1 and x_2.

Lagrangian of the system is,

$$L = T - U$$
$$= \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2\right) - \frac{1}{2}kx_1^2 - \frac{1}{2}k'(x_2 - x_1)^2 - \frac{1}{2}kx_2^2.$$

as the stretch of the middle spring is $(x_2 - x_1)$

E-L eq^n for x_1 gives:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = 0$$

$$\Rightarrow \boxed{m\ddot{x}_1 = -kx_1 + k'(x_2 - x_1)} \quad \text{——— (i)}$$

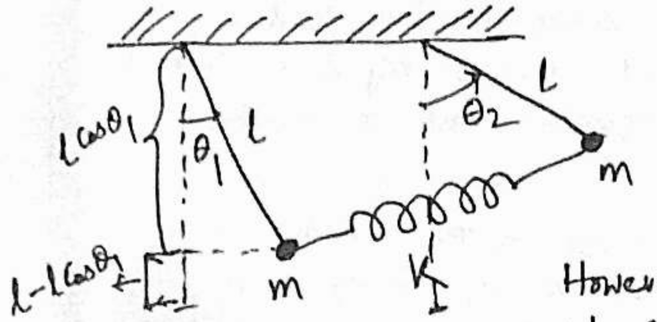
E-L eq^n for x_2 gives:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} - \frac{\partial L}{\partial x_2} = 0$$

$$\Rightarrow \boxed{m\ddot{x}_2 = -kx_2 - k'(x_2 - x_1)} \quad \text{——— (ii)}$$

Eq^n's (i), (ii) are coupled differential eq^n's. In order to solve those to find $x_1(t)$ and $x_2(t)$, we can use the "matrix method."

Example 10: Coupled pendulum :



Assume, both pendulum have mass m and length l. The generalized coordinates will be $r_1, \theta_1, r_2, \theta_2$. However, there are two holonomic constraint eq^{n}s :

$$q_1 = r_1 - l = 0$$
$$q_2 = r_2 - l = 0$$

So, the reduced generalized coordinates will be θ_1 & θ_2.

Kinetic energy will be

$$T = \frac{1}{2} m l^2 \left(\dot{\theta}_1^2 + \dot{\theta}_2^2 \right)$$

The potential energy of the system is the sum of the potential energy associated with the change in the height of the masses and the potential energy associated with the extension or compression of the spring.

The total potential energy will be

$$U = mg(l - l\cos\theta_1) + mg(l - l\cos\theta_2) + \frac{1}{2}K(l\sin\theta_1 - l\sin\theta_2)^2$$

Now use the "small angle approximation",

$$\cos\theta_1 \sim 1 - \frac{1}{2}\theta_1^2$$
$$\cos\theta_2 \sim 1 - \frac{1}{2}\theta_2^2$$
$$\sin\theta_1 \sim \theta_1$$
$$\sin\theta_2 \sim \theta_2$$

ignoring terms with $\mathcal{O}(\theta^3)$ and higher.

So, $$U \approx mgl\left[1 - \left(1 - \frac{1}{2}\theta_1^2\right)\right] + mgl\left[1 - \left(1 - \frac{1}{2}\theta_2^2\right)\right] + \frac{1}{2}kl^2(\theta_1 - \theta_2)^2$$

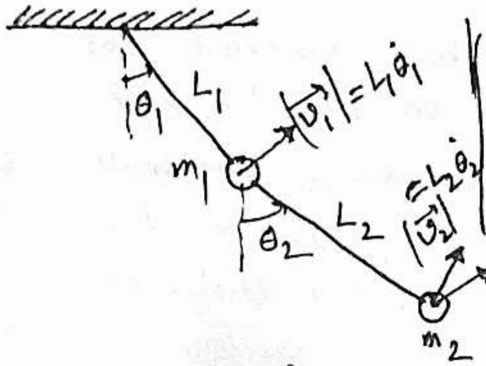
$$= \frac{1}{2}mgl\left(\theta_1^2 + \theta_2^2\right) + \frac{1}{2}kl^2(\theta_1 - \theta_2)^2$$

So, the Lagrangian,

$$L = \frac{1}{2}ml^2\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right) - \frac{1}{2}mgl\left(\theta_1^2 + \theta_2^2\right) - \frac{1}{2}kl^2(\theta_1 - \theta_2)^2$$

$\longrightarrow$ Construct the E-L eq^n (Home Work!)

Example 11: Double pendulum in series:



Consider a series-coupled double pendulum of masses m_1 & m_2 connected by rigid massless rods of lengths L_1 & L_2.

For this system, the kinetic energy of the masses m_1 and m_2 are:

$$T_1 = \frac{1}{2} m_1 L_1^2 \dot{\theta}_1^2$$

$$T_2 = \frac{1}{2} m_2 \left[L_1^2 \dot{\theta}_1^2 + 2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + L_2^2 \dot{\theta}_2^2 \right]$$

Note that, the velocity of m_2 is the vector sum of the two velocities $\vec{v}_1$ and $\vec{v}_2$, where $|\vec{v}_1| = L_1 \dot{\theta}_1$ and $|\vec{v}_2| = L_2 \dot{\theta}_2$, seperated by the angle $(\theta_2 - \theta_1)$.

Thus the total kinetic energy is

$$T = \frac{1}{2}(m_1 + m_2) L_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 L_2^2 \dot{\theta}_2^2 + m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

The total potential energy is

$$U = m_1 g L_1 (1 - \cos\theta_1) + m_2 g L_1 (1 - \cos\theta_1) + m_2 g L_2 (1 - \cos\theta_2)$$

$$= (m_1 + m_2) g L_1 (1 - \cos\theta_1) + m_2 g L_2 (1 - \cos\theta_2)$$

Assuming the small-angle approximation, $\cos\theta \sim 1 - \frac{1}{2}\theta^2$ and ignoring all terms with $\mathcal{O}(\theta^3)$ and above, we get

$$L = \frac{1}{2}(m_1 + m_2) L_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 L_2^2 \dot{\theta}_2^2 + m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2$$

$$- \frac{1}{2}(m_1 + m_2) g L_1 \theta_1^2 - \frac{1}{2} m_2 g L_2 \theta_2^2$$

here putting $\cos(\theta_1 - \theta_2) \sim 1$ as there is a term $\dot{\theta}_1 \dot{\theta}_2$, so ignoring $\mathcal{O}(\theta^3)$

Find out the E-L eq^{n}s for θ_1 & θ_2.

(Home work!)