Chapter 1 : Basic Concepts of Differential Equations

1 Differential Equations :

A differential equation is an equation involving an unknown function and its derivatives.

Example :

(a) Newton's law : Mass times acceleration equals force, $m\vec{a} = \vec{F}$, which can be written as :

$$m \frac{d^2\vec{x}}{dt^2}(t) = \vec{F}\left(t, \vec{x}(t), \frac{d\vec{x}}{dt}(t)\right),$$

where the unknown is $\vec{x}(t)$ — the position of the particle in space at the time t. The force $\vec{F}$ may depend on time, on the particle's position in space, and on the particle's velocity.

[Remark : This is a second order Ordinary Differential Equation (ODE).]

(b) Radioactive Decay : The amount u of a radioactive material changes in time as follows :

$$\frac{du}{dt}(t) = -k\,u(t), \quad k > 0,$$

where k is a positive constant representing radioactive properties of the material.

[Remark : This is a first order ODE.]

(c) The Heat Equation : The temperature T in a solid material changes in time and in three space dimensions — labeled by $\vec{x} = (x, y, z)$ — according to the equation :

$$\frac{\partial T}{\partial t}(t, \vec{x}) = k\left(\frac{\partial^2 T}{\partial x^2}(t,\vec{x}) + \frac{\partial^2 T}{\partial y^2}(t,\vec{x}) + \frac{\partial^2 T}{\partial z^2}(t,\vec{x})\right), \quad k > 0,$$

where k is a positive constant representing thermal properties of the material.

[Remark : This is a first order in time and second order in space Partial Differential Equation (PDE).]

(d) The Wave Equation : A wave perturbation u propagating in time t and in space $\vec{x} = (x, y, z)$ through the media with wave speed $v > 0$ will be written as:

$$\frac{\partial^2 u}{\partial t^2}(t, \vec{x}) = v^2 \left(\frac{\partial^2 u}{\partial x^2}(t, \vec{x}) + \frac{\partial^2 u}{\partial y^2}(t, \vec{x}) + \frac{\partial^2 u}{\partial z^2}(t, \vec{x}) \right).$$

[Remark: This is a second order in time and space PDE]

- Ordinary Differential Equation (ODE) — the unknown function depends on a single independent variable — Example (a) and (b).
- Partial Differential Equation (PDE) — the unknown function depends on two or more independent variables — Example (c) and (d).
- Order of differential equation — is the order of the highest derivative appearing in the equation — Example (a) - second order, Example (b) - first order, Example (c) - first order in time and second order in space variables, Example (d) - second order in time and space variables.
- Degree of differential eq^n: — is the power to which the highest order derivative is raised, after the eq^n has been rationalised to contain only integer powers of derivatives. ✱✱✱

2 Notation :

The expressions $y', y'', y''', y^{(4)}, \ldots, y^{(n)}$ are often used to represent, respectively, the first, second, third, fourth, ..., nth derivatives of the unknown function y with respect to the independent variable under consideration.

Thus, y'' represents $\frac{d^2y}{dx^2}$ if the independent variable is x. If the independent variable is time (t), primes are often replaced by dots. Thus, $\dot{y}$, $\ddot{y}$, $\dddot{y}$ represent $\frac{dy}{dt}$, $\frac{d^2y}{dt^2}$, $\frac{d^3y}{dt^3}$, respectively.

y is also called the "dependent variable".

✱✱✱ $\frac{d^3y}{dx^3} + x\left(\frac{dy}{dx}\right)^{3/2} + x^2 y = 0 \longrightarrow$ 3rd order and 2nd degree since after rationalisation $\rightarrow \left(\frac{d^3y}{dx^3}\right)^2$.

(3) Solutions :

A solution of a differential equation in the unknown function y and the independent variable x on some interval, say $\mathcal{I}$, is a function $y(x)$ that satisfies the differential equation identically for all x in that interval $\mathcal{I}$.

Example: Is $y(x) = c_1 \sin 2x + c_2 \cos 2x$ a solution of $y'' + 4y = 0$? where c_1 and c_2 are arbitrary constants.

$\Rightarrow$ Differentiating y, we find $y' = 2c_1 \cos 2x - 2c_2 \sin 2x$ and $y'' = -4c_1 \sin 2x - 4c_2 \cos 2x$. Hence, $y'' + 4y = 0$. So, $y = c_1 \sin 2x + c_2 \cos 2x$ satisfies the differential equation for all values of x, and thus y is a solution on the interval $(-\infty, +\infty)$.

Example: Determine whether $y = x^2 - 1$ is a solution of $(y')^4 + y^2 = -1$.

$\Rightarrow$ As the L.H.S. of the DE must be positive, while the R.H.S. is negative, the given DE has no solution.

$\boxed{\text{The general sol}^n \text{ of an } n^{th}\text{-order ODE contains } n \text{ arbitrary constants.}}$

Note that, we see some DEs have infinitely many solutions (e.g. $y'' + 4y = 0$, as c_1, c_2 are arbitrary), whereas other DEs have no solutions (e.g. $(y')^4 + y^2 = -1$). It is also possible that a DE has exactly one solution. For example $(y')^4 + y^2 = 0$ has only one solution $y = 0$. Thus, we define

- Particular solution : which is any one solution of the DE.
- General solution : which is the set of all solutions.

Example: The general solution of the DE $y'' + 4y = 0$ is $y = c_1 \sin 2x + c_2 \cos 2x$. That is, every particular solution has this general form. A few particular solutions are:

(i) $y = 5 \sin 2x - 3 \cos 2x$ $(c_1 = 5,\ c_2 = -3)$

(ii) $y = \sin 2x$ $(c_1 = 1,\ c_2 = 0)$

(iii) $y = 0$ $(c_1 = c_2 = 0)$.

(*) The general solution of a DE cannot always be expressed by single formula — e.g. $y' + y^2 = 0$ has two particular solutions: $y = 1/x$ and $y = 0$.

4 Initial-Value and Boundary-Value Problems:

A differential equation along with some conditions on the unknown function (y) and its derivatives (y', y''.. etc.), all given at the same value of the independent variable (say, at $x=0$, $y=0$, $y'=1$), constitutes an initial-value problem and the conditions are known as initial conditions.

If the conditions are given at more than one value of the independent variable (say, at $x=0$, $y=y'=0$ and at $x=2\pi$, $y=y'=\infty$), then the problem is a boundary-value problem and the conditions are boundary conditions.

Example: The DE $y''+2y'=e^x$ with $y(\pi)=1$, $y'(\pi)=2$ is an initial-value problem, whereas $y''+2y'=e^x$ with $y(0)=1$, $y(1)=1$ is a boundary-value problem.

5 Linear differential equations:

Consider a differential equation (first order) in the standard form:

$$y' = f(x,y)$$

If $f(x,y)$ can be written as $f(x,y) = -p(x)y + q(x)$ (that is, as a function of x times y, plus another function of x), then the differential equation is linear. First order linear differential equations can always be expressed as:

$$y' + p(x)y = q(x)$$

Note that, p, q can also be constants for some cases.

Example: Few examples of 1^{st} order linear ODE are:

$y' = 2y+3$, where $p(x) = -2$ and $q(x) = 3$.

$y' = -\frac{2}{t}y + 4t$ where $p(t) = 2/t$ and $q(t) = 4t$.

Whereas, the equation $y' = -\frac{2}{ty} + 4t$ is nonlinear.

For higher order ODE, it's said to be linear if each term in it is such that the dependent variable (y) or its higher derivatives occur only once, and only to the first power.

Thus $\frac{d^3y}{dx^3} + y\frac{dy}{dx} = 0$ is not linear (as y and dy/dx occur together), but

$$x^3\frac{d^3y}{dx^3} + e^x \sin x \frac{dy}{dx} + y = \ln x$$ is linear.

Similarly, $\frac{d^3y}{dx^3} + \frac{dy}{dx} + y = 0$ is linear, but

$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^2 + y = 0$ is not linear, as $\frac{dy}{dx}$ appears with power of 2.

Few examples of non-linear ODE are:

$y' + \frac{1}{y} = 0$ is non-linear, as $\frac{1}{y}$ is not a first power.

$y' + y^2 = 0$ is non-linear, as y^2 is not a first power.

$y'' + \sin(y) = 0$ is non-linear, as $\sin(y)$ is not a first power.

$y'y = 1$ is non-linear, as y and y' occur together.

- Bernoulli Equation:

A Bernoulli differential equation has a form

$y' + p(x)y = q(x)y^n$, where n denotes a real number.

When $n = 0$ or $n = 1$, a ~~Bern~~ Bernoulli equation reduces to a linear equation, otherwise, it's a non-linear eq^n.
Later, we'll solve these equations and see how these can be made linear eq^{n}s by substitution.

Chapter 2: Solutions of First-order Differential Equations

A) First-degree first-order equations:

These equations contains only $\frac{dy}{dx}$ equated to some function of x and y.
They can be written as either of two equivalent standard form:

$$\frac{dy}{dx} = F(x,y), \qquad A(x,y)\,dx + B(x,y)\,dy = 0$$

where, $F(x,y) = -\dfrac{A(x,y)}{B(x,y)}$ and $F(x,y)$, $A(x,y)$, $B(x,y)$ are in general functions of both x and y. Which of these two forms is more useful depends on the type of equation being considered.

There are several different types of first-degree first-order ODEs which are discussed below.

1) Separable-variable equations:

These equations can be written in form like

$$\boxed{\frac{dy}{dx} = f(x)\,g(y)}$$

, including cases in which $f(x)$ or $g(x)$ is simply a const.

Their solutions can be obtained at once by integrating:

$$\int \frac{dy}{g(y)} = \int f(x)\,dx$$ ~~+ I.C. (integration constant)~~

Ex1: Solve, $\dfrac{dy}{dx} = x + xy$

solⁿ:

$$\frac{dy}{dx} = x(1+y)$$

$$\Rightarrow \int \frac{dy}{1+y} = \int x\,dx$$

$$\Rightarrow \ln(1+y) = \frac{x^2}{2} + c \quad (c = \text{integration constant})$$

So, $$1+y = \exp\left(\frac{x^2}{2} + c\right)$$

$$\Rightarrow y = A\exp(x^2/2)$$, where A is another constant.

• Sometimes when the variables are not separable, the DE may be reduced to one in which they are separable by a change of variable.

$$\boxed{\frac{dy}{dx} = F(ax + by)}$$

let $w = ax + by$

$$\Rightarrow dw = a\,dx + b\,dy$$

$$\Rightarrow b\frac{dy}{dx} = \frac{dw}{dx} - a$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{b}\left(\frac{dw}{dx} - a\right) = F(w)$$

$$\Rightarrow \frac{dw}{dx} = bF(w) + a$$

$$\Rightarrow \frac{dw}{a + bF(w)} = dx.$$

Ex2: Solve, $\frac{dy}{dx} = 8x + 4y + (2x+y-1)^2$

Let, $w = 2x + y$

$$\Rightarrow \frac{dw}{dx} = 2 + \frac{dy}{dx}.$$

$$\Rightarrow \frac{dy}{dx} = \frac{dw}{dx} - 2 = 4w + (w-1)^2$$

$$\Rightarrow \frac{dw}{dx} = 4w + (w-1)^2 + 2$$

$$\Rightarrow \int \frac{dw}{w^2 + 2w + 3} = \int dx$$ → which can be solved. as the variables are separated.

2 Exact equations :

An exact first-degree first-order ODE is one of the form :

$$\boxed{A(x,y)\,dx + B(x,y)\,dy = 0} \text{ and for which } \boxed{\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}}$$

$A(x,y)\,dx + B(x,y)\,dy$ is known as exact differential $dU(x,y)$.

$$A\,dx + B\,dy = dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy$$

So, $A(x,y) = \dfrac{\partial U}{\partial x}$ —— (i)

$B(x,y) = \dfrac{\partial U}{\partial y}$ —— (ii)

Since, $\dfrac{\partial^2 U}{\partial x \partial y} = \dfrac{\partial^2 U}{\partial y \partial x}$, we therefore require

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}.$$

Then we can write $A(x,y)\,dx + B(x,y)\,dy = dU(x,y) = 0$

which has a solution, $U(x,y) = C$ (const.) and $U(x,y)$ is given by, from eq^n (i),

$$U(x,y) = \int A(x,y)\,dx + F(y) \text{ —— (iii)}$$ [as U can be fn. of x and y in general]

$F(y)$ can be obtained by partially differentiating eq^n (iii) w.r.t. y and equating that $\dfrac{\partial U}{\partial y}$ to that found from eq^n (ii).

Ex1: Solve
$$x\frac{dy}{dx} + 3x + y = 0$$

sol^n:

Rearranging it into the form of $A(x,y)\,dx + B(x,y)\,dy = 0$, we have

$$x\,dy + 3x\,dx + y\,dx = 0$$

$$\Rightarrow (3x+y)\,dx + x\,dy = 0$$

i.e. $A(x,y) = 3x + y$ and $B(x,y) = x$

Since $\dfrac{\partial A}{\partial y} = 1 = \dfrac{\partial B}{\partial x}$, this eq^n is exact.

So, $(3x+y)\,dx + x\,dy = dU(x,y) = 0$

$\Rightarrow U(x,y) = \int (3x+y)\,dx + F(y) = \text{Const.}$ (using eq^n (iii))

(as $dU = 0 \Rightarrow U = \text{Const.}$)

$$\Rightarrow \frac{3x^2}{2} + yx + F(y) = U(x,y) = \text{Const. (say } c_1)$$

Partially differentiating this w.r.t. y gives:

$$x + \frac{dF(y)}{dy} = \frac{\partial U}{\partial y}$$ [as F is fn. of only y, $\frac{\partial F}{\partial y}$ means $\frac{dF}{dy}$.]

But, from eqn (ii), $B(x,y) = \frac{\partial U}{\partial y} = x$.

So, $x + \frac{dF}{dy} = x$

$$\Rightarrow \frac{dF}{dy} = 0$$

$$\Rightarrow F = \text{constant. (say, } c_2)$$

So, $\frac{3x^2}{2} + yx + c_2 = c_1$

$$\Rightarrow \frac{3x^2}{2} + yx = c \quad (\text{where } c = c_1 - c_2)$$ Thus, $y = \frac{c - 3x^2/2}{x}$.

<u>Ex 2.</u> Solve, $2xy\,dx + (1+x^2)dy = 0$

Here, $A(x,y) = 2xy$, $B(x,y) = 1+x^2$

$$\frac{\partial A}{\partial y} = 2x = \frac{\partial B}{\partial x}$$, so the given eqn is exact.

So, $2xy\,dx + (1+x^2)\,dy = dU(x,y) = 0$

or, $U(x,y) = \text{Const} = c_1$

Now, $dU = A\,dx + B\,dy \Rightarrow A = \frac{\partial U}{\partial x}$ and $B = \frac{\partial U}{\partial y}$

$$\Rightarrow U(x,y) = \int A\,dx + F(y)$$

$$= 2\int xy\,dx + F(y)$$

$$\Rightarrow U(x,y) = x^2 y + F(y)$$

Partially differentiating w.r.t. y, we get

$$\frac{\partial U}{\partial y} = x^2 + \frac{dF}{dy} = B(x,y) = 1+x^2$$

$$\Rightarrow \frac{dF}{dy} = 1 \Rightarrow F(y) = y + c_2$$ (c_2 is another const.)

Thus, from, $x^2 y + F(y) = c_1$, we can now write,

$$x^2 y + y = c \quad (\text{where, } c = c_1 - c_2)$$

$$\Rightarrow y = \frac{c}{1+x^2}.$$

3 Inexact equations : integrating factors :

Differential equations that may be written in the form $\boxed{A(x,y)\,dx + B(x,y)\,dy = 0}$ but for which $\boxed{\frac{\partial A}{\partial y} \neq \frac{\partial B}{\partial x}}$ are known as inexact equations.

However, the differential $A\,dx + B\,dy$ can always be made exact by multiplying by an "integrating factor" $\mu(x,y)$, which obeys

$$\frac{\partial(\mu A)}{\partial y} = \frac{\partial(\mu B)}{\partial x}. \qquad \text{—— (iv)}$$

➡ If the integrating factor is a function of both x and y, i.e. $\mu = \mu(x,y)$, there exists no general method for finding it; however it may sometimes be found by inspection.

➡ If the integrating factor is a function of either x or y alone, then eq^n (iv) can be solved to find it.

(a) For example, if $\mu = \mu(x)$, then from eq^n (iv),

$$\mu \frac{\partial A}{\partial y} = \mu \frac{\partial B}{\partial x} + B \frac{d\mu}{dx}$$

$$\Rightarrow \frac{B}{\mu}\frac{d\mu}{dx} = \frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}$$

$$\Rightarrow \frac{d\mu}{\mu} = \frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right)dx = f(x)\,dx,$$

Note that, here we require $f(x)$ also be a function of x only. This, in fact, provides a general method of determining whether μ is a function of x alone or not. This integrating factor is then given by

$$~~\mu(x)~~ \quad \int \frac{d\mu}{\mu} = \int f(x)\,dx$$

$$\Rightarrow \mu(x) = \exp\left\{\int f(x)\,dx\right\}$$

where, $f(x) = \frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right)$.

(b) Similarly, if $\mu = \mu(y)$ then

$$\mu(y) = \exp\left\{ \int g(y)\, dy \right\}, \quad \text{where } g(y) = \frac{1}{A}\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right)$$

<u>Ex.1.</u> Solve $\quad y\,dx - x\,dy = 0$

<u>Sol^n</u>: Comparing this eq^n with $A(x,y)\,dx + B(x,y)\,dy = 0$, we get,

$$A(x,y) = y \quad \text{and} \quad B(x,y) = -x$$

$$\frac{\partial A}{\partial y} = 1 \qquad \frac{\partial B}{\partial x} = -1$$

So, the given ODE is <u>not exact</u> in its present form, we need an integrating factor to make it exact.

Now, try to find if the integrating factor is function of either x or y alone, and for that we compute

$$\frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right).$$

$$= -\frac{1}{x}(1+1) = -\frac{2}{x}, \text{ which is a function of } x \text{ alone.}$$

So, an integrating factor exists which is a function of x alone.

$$\mu(x) = \exp\left\{ \int f(x)\, dx \right\}$$

$$= \exp\left\{ -2\int \frac{dx}{x} \right\} = \exp(-2\ln x) \text{ [here we ignore the arbitrary const. of integration]}$$

$$= -\frac{1}{x^2}.$$

Multiplying the given DE by $-\frac{1}{x^2}$, we get

$$-\frac{y}{x^2}\,dx + \frac{1}{x}\,dy = 0$$

Now, $A(x,y) = -y/x^2 \qquad B(x,y) = \frac{1}{x}.$

$$\frac{\partial A}{\partial y} = -\frac{1}{x^2}, \qquad \frac{\partial B}{\partial x} = -\frac{1}{x^2}.$$

So, now the eq^n is exact.

$$-\frac{y}{x^2}\,dx + \frac{1}{x}\,dy = dU(x,y) = 0$$

$$\Rightarrow U(x,y) = \text{const.} = C_1$$

We know for exact DE, $dU = A\,dx + B\,dy$

$$\Rightarrow A = \frac{\partial U}{\partial x} \quad \& \quad B = \frac{\partial U}{\partial y}.$$

Then, $U(x,y) = \int A\,dx + F(y)$

$$= -\int \frac{y}{x^2}\,dx + F(y)$$

$$= +\frac{y}{x} + F(y)$$

Partially differentiating w.r.t. y, we get

$$\frac{\partial U}{\partial y} = \frac{1}{x} + \frac{dF}{dy} = B(x,y) = \frac{1}{x}.$$

$$\Rightarrow \frac{dF}{dy} = 0$$

$$\Rightarrow F = \text{Const.} = C_2$$

So, from $\frac{y}{x} + F(y) = U(x,y) = C_1$, we get

$$\Rightarrow \frac{y}{x} + C_2 = C_1$$

$$\Rightarrow y = cx \ (\text{where } c = C_1 - C_2).$$

The same solⁿ can also be obtained from multiplying the DE by $-\frac{1}{x^2}$ and looking at that form as:

$$-\frac{y\,dx - x\,dy}{x^2} = 0$$

$$\Rightarrow \frac{x\,dy - y\,dx}{x^2} = 0$$

$$\Rightarrow d(y/x) = 0$$

$$\Rightarrow y/x = \text{Const.} = c$$

$$\Rightarrow y = cx.$$

Ex2: Try yourself:
Solve $\frac{dy}{dx} = -\frac{2}{y} - \frac{3y}{2x}$.

[Ans: $\mu(x) = x^2$
and solⁿ is $x^4 + y^2x^3 = c$]

Ex2: Try yourself: Solve $x\frac{dy}{dx} + 2y + x^2 = 0$
[Ans: $\mu(x) = x$ and the solⁿ is: $y = \frac{c - x^4}{4x^2}$]
Actually, this is a linear ODE, which will be solved later.

(4) Linear equations :

Linear first-order ODEs are a special case of inexact ODEs, and can be written in the conventional form:

$$\boxed{\frac{dy}{dx} + p(x)y = q(x)}$$

In this case, the integrating factor is always a function of x alone, i.e., $\mu = \mu(x)$. It can be shown that the integrating factor has a simple form:

$$\boxed{\mu(x) = \exp\left\{ \int p(x)\,dx \right\}}$$

Ex. 1: Solve $\dfrac{dy}{dx} + 2xy = 4x$

Sol^n: The eq^n is linear which has a form like

$$\frac{dy}{dx} + P(x)y = q(x)$$

where $p(x) = 2x$ and $q(x) = 4x$

The integrating factor $\mu(x) = \exp\left\{\int 2x\,dx\right\} = e^{x^2}$

Multiplying the ODE by e^{x^2}, and then integrating we get,

$$e^{x^2}\frac{dy}{dx} + e^{x^2}.2xy = 4x.e^{x^2}$$

$$\Rightarrow \int d\left(e^{x^2}.y\right) = \int 4xe^{x^2}dx$$

$$\Rightarrow ye^{x^2} = 4\int xe^{x^2}dx = 4.\frac{1}{2}\int d\left(e^{x^2}\right) = 2e^{x^2} + c$$

So, the sol^n, $\underline{y = 2 + ce^{-x^2}}$.

Ex 2:

$$x\frac{dy}{dx} + 2y + x^2 = 0$$

Sol^n: Rearranging the eq^n we get:

$$\frac{dy}{dx} + \frac{2}{x}y = -x$$

Which is a linear ODE, with form $\dfrac{dy}{dx} + p(x)y = q(x)$

where, $p(x) = \frac{2}{x}$, $q(x) = -x$

So, the integrating factor will be

$$\mu(x) = \exp\left\{\int p(x)\,dx\right\}$$

$$= \exp\left\{2\int \frac{dx}{x}\right\} = \exp(\ln x^2) = x^2.$$

Multiplying the ~~given~~ rearranged eq^n by x^2, we obtain

$$x^2 \frac{dy}{dx} + 2xy = -x^3$$

$$\Rightarrow \int d(x^2 y) = -\int x^3\,dx$$

$$\Rightarrow x^2 y = -\frac{x^4}{4} + C$$

$$\Rightarrow y = \frac{C - x^4/4}{x^2} = \frac{C' - x^4}{4x^2}.$$

Ex3: Solve $y' + \frac{4}{x} y = x^4$

Sol^n This is a linear ODE of the form $\frac{dy}{dx} + p(x)y = q(x)$

where $p(x) = \frac{4}{x}$ and $q(x) = x^4$.

So, the integrating factor is

$$\mu(x) = \exp\left\{\int p(x)\,dx\right\}$$

$$= \exp\left\{\int 4\frac{dx}{x}\right\} = \exp(\ln x^4) = x^4.$$

Multiplying the DE by x^4 and then integrating, we get

$$x^4 y' + 4x^3 y = x^8$$

$$\Rightarrow \frac{d}{dx}(yx^4) = x^8$$

$$\Rightarrow \int d(yx^4) = \int x^8\,dx$$

$$\Rightarrow yx^4 = \frac{x^9}{9} + C$$

$$\Rightarrow y = \frac{C}{x^4} + \frac{1}{9}x^5.$$

5 Homogeneous equations: defn: If each term of a differential eqn contains $y(x)$ or its derivative.

These equations have a form like

$$\boxed{\frac{dy}{dx} = F(x,y)} \text{ having the property } \boxed{F(\lambda x, \lambda y) = F(x,y)}.$$

Then the RHS of the homogeneous eqn can be written as a fn. of $\frac{y}{x}$, i.e. $F(x,y) = f(y/x)$. The eqn can be solved by making the substitution $\boxed{y = vx,}$ so that,

$$\frac{dy}{dx} = v + x\frac{dv}{dx} = F(v)$$

This is now a separable eqn and can be integrated directly to give

$$\int \frac{dv}{F(v)-v} = \int \frac{dx}{x}$$

Ex.1: Solve $\qquad y' = \frac{y+x}{x}$

Soln: This eqn is not separable in its current form, it has the form $y' = F(x,y)$ with $F(x,y) = \frac{y+x}{x}$

where $\quad F(\lambda x, \lambda y) = \frac{\lambda y + \lambda x}{\lambda x} = \frac{y+x}{x} = F(x,y)$

So, the eqn is homogeneous. Substituting $y = vx$ we obtain

$$y' = v + x\frac{dv}{dx} = \frac{vx+x}{x} = v+1$$

$$\Rightarrow x\frac{dv}{dx} = 1$$

$$\Rightarrow \int dv = \int \frac{dx}{x}$$

$$\Rightarrow \underline{v = \ln x + c} = \ln(kx) \quad (\text{where } c = \ln k)$$

$$\Rightarrow \underline{y = x\ln(kx)}$$

Ex 2: Solve $\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$

Sol^n: This is a homogeneous eq^n. Substituting $y = vx$ we obtain

$$\frac{dy}{dx} = v + x\frac{dv}{dx} = v + \tan v$$

$$\Rightarrow x\frac{dv}{dx} = \tan v$$

$$\Rightarrow \int \cot v \, dv = \int \frac{dx}{x}.$$

$$\Rightarrow \int \frac{\cos v}{\sin v} dv = \ln x + c_1$$

$$\Rightarrow \int \frac{d(\sin v)}{\sin v} = \ln x + c_1$$

$$\Rightarrow \ln(\sin v) = \ln x + \ln c = \ln(cx)$$

$$\Rightarrow \sin v = cx$$

$$\Rightarrow v = \sin^{-1} cx \quad \text{or,} \quad \underline{y = x\sin^{-1} cx}.$$

6 Bernoulli's Equation:

Bernoulli's differential eq^n has the form

$$\boxed{\frac{dy}{dx} + p(x)y = q(x)\,y^n}, \quad \text{where } n \text{ is a real number.}$$

For $n = 0$ or 1, this becomes linear eq^n. However for other values of n, it is non-linear due to the extra y^n factor on the RHS.

This eq^n can be made linear by substituting

$$\boxed{v = y^{1-n}} \quad \Rightarrow y = y^n v$$

So that, $\frac{dv}{dx} = (1-n)\,y^{-n} \cdot \frac{dy}{dx}$.

$$\Rightarrow \frac{dy}{dx} = \left(\frac{y^n}{1-n}\right)\frac{dv}{dx}.$$

Substituting this into the original eq^n

$$\left(\frac{y^n}{1-n}\right)\frac{dv}{dx} + p(x)\,y^n v = q(x)\,y^n$$

$$\Rightarrow \boxed{\frac{dv}{dx} + (1-n)\,p(x)\,v = (1-n)\,q(x)}$$

which is a linear eq^n and can be solved by integrating factor.

Ex1: Solve
$$y' + xy = xy^2$$

Sol^n: This eq^n is not linear. It is a Bernoulli's eq^n having the form
$$\frac{dy}{dx} + p(x)\,y = q(x)\,y^n$$
with $p(x) = x$, $q(x) = x$ and $n = 2$.

Thus substitute
$$v = y^{1-n} = y^{-1}$$
$$\Rightarrow y = 1/v$$
$$\Rightarrow \frac{dy}{dx} = -\frac{1}{v^2}\frac{dv}{dx}.$$

Substituting this into the original eq^n we obtain,
$$-\frac{1}{v^2}\frac{dv}{dx} + \frac{1}{v}x = \frac{1}{v^2}x$$
$$\Rightarrow \frac{dv}{dx} - vx = -x \quad \text{———} \quad (1)$$

which is a linear eq^n for the unknown fn. $v(x)$, it has the form of
$$\frac{dv}{dx} + p(x)\,v = q(x)$$
where $p(x) = -x$, $q(x) = -x$.

The integrating factor will be
$$\mu(x) = \exp\left\{\int p(x)\,dx\right\}$$
$$= \exp\left\{\int (-x)\,dx\right\} = e^{-x^2/2}.$$

Multiplying the eq^n (1) by $e^{-x^2/2}$ we obtain,
$$e^{-x^2/2}\frac{dv}{dx} - x e^{-x^2/2} v = -x e^{-x^2/2}$$
$$\Rightarrow \frac{d}{dx}\left(v e^{-x^2/2}\right) = -x e^{-x^2/2}$$
$$\Rightarrow \int d\left(v e^{-x^2/2}\right) = -\int x e^{-x^2/2}\,dx = \int d\left(e^{-x^2/2}\right)$$
$$\Rightarrow v e^{-x^2/2} = e^{-x^2/2} + c$$

$$\Rightarrow v(x) = ce^{x^2/2} + 1$$

So, the sol^n of the original differential eq^n will be

$$y = \frac{1}{v(x)} = \frac{1}{ce^{x^2/2}+1}.$$

Ex2:

Solve $$\frac{dy}{dx} + \frac{y}{x} = 2x^3 y^4.$$

Sol^n: This is a Bernoulli's eq^n having the form

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

with $p(x) = \frac{1}{x}$, $q(x) = 2x^3$ and $n = 4$.

Thus substitute $v = y^{1-4} = y^{-3} \longrightarrow y = v^3$

$$\Rightarrow \frac{dv}{dx} = -3y^{-4}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y^4}{3}\frac{dv}{dx}.$$

Substituting this into the ODE and rearranging, we obtain

$$-\frac{y^4}{3}\frac{dv}{dx} + \frac{y}{x} = 2x^3y^4$$

$$\Rightarrow -\frac{1}{3}\frac{dv}{dx} + y^{-3}/x = 2x^3$$

$$\Rightarrow -\frac{1}{3}\frac{dv}{dx} + \frac{v}{x} = 2x^3$$

$$\Rightarrow \frac{dv}{dx} - \frac{3v}{x} = -6x^3 \quad \text{——— (i)}$$

which is a linear eq^n of the form $\frac{dv}{dx} + p(x)v = q(x)$

where $p(x) = -\frac{3}{x}$, $q(x) = -6x^3$

So, the integrating factor will be

$$\mu(x) = \exp\left\{\int p(x)\,dx\right\}$$

$$= \exp\left\{-3\int\frac{dx}{x}\right\} = \exp(-3\ln x) = \frac{1}{x^3}.$$

Multiplying eqⁿ ① by $1/x^3$, we get

$$\frac{1}{x^3}\frac{dv}{dx} - \frac{3}{x^4}v = -6$$

$$\Rightarrow \frac{d}{dx}\left(\frac{v}{x^3}\right) = -6$$

So, $$\int d\left(\frac{v}{x^3}\right) = -6\int dx$$

$$\Rightarrow \frac{v}{x^3} = -6x + c$$

$$\Rightarrow y^{-3} = -6x^4 + Cx^3 \quad (\text{as } v = y^{-3})$$

$$\Rightarrow \underline{y(x) = \left(Cx^3 - 6x^4\right)^{3}}.$$

(B) Higher-degree first-order equations :

These type of equations doesnot occur often in the description of physical systems. They can be written as $F\left(x, y, \frac{dy}{dx}\right) = 0$. The most general standard form is

$$\boxed{\left(\frac{dy}{dx}\right)^n + a_{n-1}(x,y)\left(\frac{dy}{dx}\right)^{n-1} + \cdots + a_1(x,y)\frac{dy}{dx} + a_0(x,y) = 0}$$

Their solutions are little complicated, and thus are beyond the scope of this lecture note.

Chapter 3 : Higher-order Ordinary Differential Equations

General form of a linear ODE of order n is

$$\boxed{a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x)} \quad (1)$$

- If the RHS, $F(x)=0$, then the equation is called "homogeneous"; otherwise it is "inhomogeneous".
- A linear ODE has "constant coefficients" if all the coefficients $a_i(x)$ in eq^n (1) are constants; if one or more of these coefficients is not constants, eq^n (1) has "variable coefficients".
- The first-order linear eq^n, studied in last chapter, is a special case of eq^n (1), where

$$a_1(x)\frac{dy}{dx} + a_0(x)y = F(x)$$

$$\Rightarrow \quad \frac{dy}{dx} + p(x)y = q(x)$$

with $p(x) = \dfrac{a_0(x)}{a_1(x)}$ and $q(x) = \dfrac{F(x)}{a_1(x)}$.

> Note that, for $F(x)=0$ (i.e. homogeneous eq^n), ~~it looks like~~ as each term contains $y(x)$ or its derivative.
>
> $$\cancel{\frac{dy}{dx} = -\frac{a_0(x)}{a_1(x)}y = F(x,y)}$$
>
> ~~where $F(x,y)$ having the property that $F(\lambda x, \lambda y) = F(x,y)$ otherwise the eq^n would not be linear~~.

- As discussed earlier, the general solution to eq^n (1) will contain n arbitrary constants, which can be determined if n boundary conditions are provided.
- Sometimes, eq^n (1) is also expressed as

$$\left(a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x)\right)y = F(x)$$

where $D \equiv \dfrac{d}{dx}$.

It's important to note that the symbol D is meaningless unless applied to a function of x and is therefore not a mathematical quantity in the usual sense. D is an operator.

① Solution of a general linear ODE of higher order :

Let's first define "complementary equation" — the eq^n formed by setting $F(x) = 0$ (i.e. homogeneous eq^n):

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)\,y = 0 \quad (2)$$

In order to solve any general eq^n of the form given in eq^n (1), we must find the general solution of this complementary eq^n i.e eq^n (2). And to determine that, we must find n linearly independent functions that satisfy it. If those linearly independent functions are $y_1(x), y_2(x), \ldots, y_n(x)$ [remember that, all are the sol^n of eq^n (2)], then the general sol^n is given by the linear superposition of these functions:

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

where C_i's are arbitrary constants that can be determined if n boundary conditions are provided. The function $y_c(x)$ is called the "complementary function".

- Condition for linear independence :

For n functions $\{y_1, y_2, \ldots, y_n\}$ to be "linearly independent" over an interval, there must ~~not~~ exist ~~any~~ a set of constants $c_1, c_2, \ldots, c_n$ such that

$$c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) \neq 0$$

over the interval in question, ~~excluding~~ except for the trivial case all c_i's are zero, i.e. $c_1 = c_2 = \cdots = c_n = 0$.

$\Rightarrow$ If there exists constants $c_1, c_2, \ldots c_n$, not all zero, such that,

$$c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) = 0$$

then the set of functions $\{y_1, y_2, \ldots, y_n\}$ is "linearly dependent".

Note that, $c_1 = c_2 = \cdots = c_n = 0$ is a set of constants that always satisfies $c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) = 0$. If there exists another set of constants, not all zero, that also satisfies the above eqⁿ, then the set of functions is linearly dependent. If the only solution is $c_1 = c_2 = \cdots = c_n = 0$, then $\{ y_1(x), y_2(x), \cdots, y_n(x) \}$ is linearly independent.

Example: The set $\{x, 5x, 1, \sin x\}$ is linearly dependent on $[-1, 1]$ since there exist constants $c_1 = -5$, $c_2 = 1$, $c_3 = 0$. and $c_4 = 0$, not all zero, for that

$$-5.(x) + 1.(5x) + 0.(1) + 0.(\sin x) = 0.$$

- Another way to check whether the set of functions is linearly independent or not, is to determine the determinant called "Wronskian", which is defined as:

$$W(y_1, y_2, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

If $W(y_1, y_2, \cdots, y_n) \neq 0$, then the n functions $y_1(x), y_2(x), \cdots, y_n(x)$ are said to be linearly independent.

However, it should be noted that, even if $W(y_1, y_2, \cdots y_n) = 0$, that doesn't guarantee that the functions are linearly dependent. In that case, one must test directly whether

$$c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) \text{ is zero or not.}$$

$\Longrightarrow$ If the original eqⁿ (1) has $F(x) = 0$ (i.e. it is homogeneous) then of course the complementary function $y_c(x)$ is already the general solution.

If $F(x) \neq 0$ (inhomogeneous), then $y_c(x)$ is only one part of the solution. The general solⁿ of eqⁿ (1) is given by

$$\boxed{y(x) = y_c(x) + y_p(x)}$$

where $y_p(x)$ is called "particular integral" or "particular solution", which can be any function that satisfies eqⁿ (1), provided it's linearly independent of $y_c(x)$.

2 Solution of higher order linear ODE with constant coefficients:

If a_i's in eq^n 1 are constants, we have

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = f(x) \quad \text{--- (3)}$$

— the method of solution is same, i.e. we need to find the complementary function $y_c(x)$ and particular integral $y_p(x)$.

2.1 Finding the complementary function $y_c(x)$:

As discussed earlier, the complementary fn. must satisfy

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0 \quad \text{--- (4)}$$

and contain n arbitrary constants (c_i's).

The standard method for finding $y_c(x)$ is to try a solution of the form $y = Ae^{\lambda x}$.

Substituting this into eq^n 4 we get a polynomial eq^n in λ of order n:

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0 \quad \text{--- (5)}$$

— this is called "auxilliary equation" or "characteristic eq^n".

In general, eq^n 5 has n roots, say $\lambda_1, \lambda_2, \ldots, \lambda_n$ — some of them may be repeated and some may be complex. The three main cases are as follows:

(i) All roots are real and distinct:

It can be shown by calculating the Wronskian of the function $y = e^{\lambda_m x}$, for $m = 1$ to n, that if all λ_m are distinct, then the sol^{n}s are linearly independent. We can therefore linearly superpose them to form complementary fn:

$$y_c(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \cdots + c_n e^{\lambda_n x}.$$

(ii) Some roots are complex:

If one of the roots of eq^n 5 is complex, say $\alpha + i\beta$, then its complex conjugate $\alpha - i\beta$ is also a root, i.e. if $\lambda_1 = \alpha + i\beta$, λ_2 must be $\alpha - i\beta$. We can write,

$$y_c(x) = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} + c_3 e^{\lambda_3 x} + \cdots + c_n e^{\lambda_n x}$$

$$= e^{\alpha x}(d_1 \cos\beta x + d_2 \sin\beta x) + c_3 e^{\lambda_3 x} + \cdots + c_n e^{\lambda_n x}.$$

(iii) Some roots are repeated:

If, for example, λ_1 occurs k times $(k>1)$ as a root of eq^n (5) (the auxiliary eq^n), then the Wronskian of these sol^{n}s, having two or more identical colums, is equal to zero — thus ~~only the~~ not all n solutions are linearly independent.

Note that, by substituting $xe^{\lambda_1 x}, x^2 e^{\lambda_1 x}, \ldots, x^{k-1} e^{\lambda_1 x}$ into eq^n (4), we find that these are also solutions of eq^n (4) and their Wronskian is non-zero. Thus we can write the complementary function for one root λ_1 occuring k times, as

$$y_c(x) = (c_1 + c_2 x + \cdots + c_k x^{k-1}) e^{\lambda_1 x} + c_{k+1} e^{\lambda_{k+1} x} + c_{k+2} e^{\lambda_{k+2} x} + \cdots + c_n e^{\lambda_n x}.$$

If more than one root is repeated, the above argument can easily be extended.

- Special case: Second order linear ODE with constant coefficients:

$$a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0 \quad \text{(homogeneous)}$$

The auxiliary eq^n reads as:

$$a_2 \lambda^2 + a_1 \lambda + a_0 = 0$$

which is obtained by substituting $y = Ae^{\lambda x}$.

The above eq^n can be factored into

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$$

with λ_1 and λ_2 are the roots.

There are three cases to consider:

(i) λ_1 and λ_2 both real and distinct:

sol^n: $y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

(ii) $\lambda_1 = \alpha + i\beta$, a complex number:

The other root λ_2 must be $\lambda_2 = \alpha - i\beta$.

sol^n: $y(x) = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} = e^{\alpha x}(d_1 \cos\beta x + d_2 \sin\beta x)$

(iii) $\lambda_1 = \lambda_2$:

Two linearly independent solutions are $e^{\lambda_1 x}$ and $x e^{\lambda_1 x}$, and the general sol^n is:

$$y(x) = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}. = (c_1 + c_2 x) e^{\lambda_1 x}$$

Ex 1:

Solve $y'' - y' - 2y = 0$

Sol^n: The auxiliary eq^n is $\lambda^2 - \lambda - 2 = 0$

$$\Rightarrow \lambda^2 - 2\lambda + \lambda - 2 = 0$$

$$\Rightarrow \lambda(\lambda - 2) + 1.(\lambda - 2) = 0$$

$$\Rightarrow (\lambda - 2)(\lambda + 1) = 0$$

So, the roots are $\lambda_1 = 2$, $\lambda_2 = -1$ which are real and distinct, thus the sol^n is:

$$y(x) = c_1 e^{2x} + c_2 e^{-x}.$$

Ex2: Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$, given that $y = 1$ and $\frac{dy}{dx} = 3$ when $x = 0$.

Sol^n: The auxiliary eq^n is: $\lambda^2 - 4\lambda + 4 = 0$

$$\Rightarrow (\lambda - 2)^2 = 0$$

which means $\lambda_1 = \lambda_2 = 2$. The general sol^n is therefore,

$$y(x) = (c_1 + c_2 x) e^{2x}.$$

At $x = 0$, $y = 1 \Rightarrow c_1 = 1$

Now, $\frac{dy}{dx} = 2(c_1 + c_2 x) e^{2x} + c_2 e^{2x}$

At $x = 0$, $\frac{dy}{dx} = 3 \Rightarrow 3 = 2c_1 + c_2$

$$\Rightarrow c_2 = 1.$$

The sol^n is:

$$y(x) = (1 + x) e^{2x}.$$

Ex3: Solve $y''' - 6y'' + 2y' + 36y = 0$

Sol^n: The auxiliary eq^n is:

$$\lambda^3 - 6\lambda^2 + 2\lambda + 36 = 0$$

let's try for integer solutions, i.e looking for solutions which are integrers and divide the constant term 36 — these are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36$.

If we try $\lambda = -2 \Rightarrow -8 - 24 - 4 + 36 = 0$

So, $\lambda = -2$ is one of the roots, let's factor that out:

$$\lambda^3 - 6\lambda^2 + 2\lambda + 36 = 0$$

$$\Rightarrow \lambda^3 + 2\lambda^2 - 8\lambda^2 + 2\lambda + 36 = 0$$

$$\Rightarrow \lambda^2(\lambda+2) - 8\lambda(\lambda+2) + 18\lambda + 36 = 0$$

$$\Rightarrow \lambda^2(\lambda+2) - 8\lambda(\lambda+2) + 18(\lambda+2) = 0$$

$$\Rightarrow (\lambda+2)(\lambda^2 - 8\lambda + 18) = 0$$

So, $\lambda_1 = -2$, $\quad \lambda_2 = \dfrac{+8 + \sqrt{64-72}}{2}$, $\quad \lambda_3 = \dfrac{+8 - \sqrt{64-72}}{2}$

$$\lambda_2 = \frac{8 + 2\sqrt{-2}}{2} = 4 + i\sqrt{2}, \qquad \lambda_3 = \frac{8 - 2\sqrt{-2}}{2} = 4 - i\sqrt{2}.$$

The solution is

$$y(x) = c_1 e^{-2x} + c_2 e^{(4+i\sqrt{2})x} + c_3 e^{(4-i\sqrt{2})x}$$

$$= c_1 e^{-2x} + e^{4x}\left(d_2 \cos\sqrt{2}x + d_3 \sin\sqrt{2}x\right)$$

2.2 Finding the particular integral $y_p(x)$:

There is no general method for finding the particular integral $y_p(x)$. However, for linear ODEs with constant coefficients and a simple R.H.S. (i.e $F(x)$ in eqn ③), $y_p(x)$ can often be found by inspection or by assuming a parameterised form similar to $F(x)$. Standard trial functions are as follows:

(i) If $F(x) = ae^{rx}$, then try $y_p(x) = be^{rx}$.

(ii) If $F(x) = a_1 \sin rx + a_2 \cos rx$ (a_1 or a_2 may be zero), then try $y_p(x) = b_1 \sin rx + b_2 \cos rx$.

(iii) If $F(x) = a_0 + a_1 x + \cdots + a_N x^N$ (some a_m may be zero), then try $y_p(x) = b_0 + b_1 x + \cdots + b_N x^N$.

(iv) If $F(x)$ is the sum or product of any of the above, then try $y_p(x)$ as the sum or product of the corresponding individual trial functions.

For example, if $F(x) = ae^{\alpha x} \sin \beta x$ or $ae^{\alpha x} \cos \beta x$, then try $y_p(x) = e^{\alpha x}(b_1 \sin \beta x + b_2 \cos \beta x)$.

- It should be noted that, the above method fails if any term in the assumed trial function is also contained within the complementary function $y_c(x)$. In such case, the trial function should be multiplied by the <u>smallest integer power of x</u>, such that it will then contain no term that appears in the complementary function.

Further methods that are useful in finding the $y_p(x)$ are based on Greens functions, Variation of parameters etc., and are beyond the scope of this lecture note.

Ex 1: Solve $y'' - y' - 2y = 4x^2$

Solⁿ: The complementary function $y_c(x)$, obtained from $y'' - y' - 2y = 0$, has already been found in earlier exercise: $y_c(x) = c_1 e^{2x} + c_2 e^{-x}$.

Here $f(x) = 4x^2$, a second degree polynomial. Thus we try:
$$y_p(x) = b_0 + b_1 x + b_2 x^2.$$

$$\Rightarrow y'_p(x) = b_1 + 2b_2 x$$

$$\Rightarrow y''_p(x) = 2b_2.$$

Substituting these results into the differential eqⁿ, we get
$$2b_2 - (b_1 + 2b_2 x) - 2(b_0 + b_1 x + b_2 x^2) = 4x^2$$
$$\Rightarrow (-2b_2)x^2 + (-2b_2 - 2b_1)x + (2b_2 - b_1 - 2b_0) = 4x^2 + 0.x + 0$$

Equating the coefficients of like powers of x, we obtain

$$-2b_2 = 4 \qquad -2b_2 - 2b_1 = 0 \qquad 2b_2 - b_1 - 2b_0 = 0$$
$$\Rightarrow b_2 = -2 \qquad \Rightarrow 2b_1 = 4 \qquad \Rightarrow -4 - 2 - 2b_0 = 0$$
$$\Rightarrow b_1 = 2 \qquad \Rightarrow 2b_0 = -6$$
$$\Rightarrow b_0 = -3.$$

So, we get, $b_0 = -3$, $b_1 = 2$, $b_2 = -2$. Hence the ~~gen~~ perticular integral is,
$$y_p(x) = -3 + 2x - 2x^2.$$

The general solⁿ is:
$$y = y_c(x) + y_p(x)$$
$$= c_1 e^{2x} + c_2 e^{-x} - 2x^2 + 2x - 3.$$

Ex 2: Solve $y'' - y' - 2y = \sin 2x$

Solⁿ: Again, $y_c(x) = c_1 e^{2x} + c_2 e^{-x}$.

Here $F(x) = \sin 2x$, thus we try

$$y_p(x) = b_1 \sin 2x + b_2 \cos 2x$$

$$\Rightarrow y'_p(x) = 2b_1 \cos 2x - 2b_2 \sin 2x$$

$$\Rightarrow y''_p(x) = -4b_1 \sin 2x - 4b_2 \cos 2x.$$

Substituting these into the differential eqⁿ, we get

$$(-4b_1 \sin 2x - 4b_2 \cos 2x) - (2b_1 \cos 2x - 2b_2 \sin 2x) - 2(b_1 \sin 2x + b_2 \cos 2x) = \sin 2x.$$

$$\Rightarrow (-6b_1 + 2b_2) \sin 2x + (-6b_2 - 2b_1) \cos 2x = (1).\sin 2x + (0).\cos 2x.$$

Equating coefficients of like terms, we obtain

$$-6b_1 + 2b_2 = 1 \qquad -6b_2 - 2b_1 = 0$$

$$\Rightarrow b_1 = -3b_2$$

putting $b_1 = -3b_2$ into $-6b_1 + 2b_2 = 1$, we get

$$18b_2 + 2b_2 = 1$$

$$\Rightarrow b_2 = \frac{1}{20} \quad \text{and} \quad b_1 = -3b_2 = -\frac{3}{20}.$$

So, $$y_p(x) = -\frac{3}{20} \sin 2x + \frac{1}{20} \cos 2x.$$

and the general solution is:

$$y(x) = y_c(x) + y_p(x)$$

$$= c_1 e^{2x} + c_2 e^{-x} - \frac{3}{20} \sin 2x + \frac{1}{20} \cos 2x.$$

Ex 3: Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$.

Sol^n: First, we need to find the complementary function $y_c(x)$.
For that, substitute $y = Ae^{\lambda x}$ and find the ~~ob~~ auxiliary eq^n of $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$, as:

$$\lambda^2 - 2\lambda + 1 = 0$$

$$\Rightarrow (\lambda - 1)^2 = 0$$

as the root $\lambda = 1$ occurs twice, we can write

$$y_c(x) = (c_1 + c_2 x)e^x.$$

Now, find the particular integral $y_p(x)$.
As, $f(x) = e^x$, our first guess at a trial particular integral would be $y_p(x) = be^x$.
But, the $y_c(x) = (c_1 + c_2 x)e^x$. We see that, e^x is already contained in it, as indeed is xe^x. Therefore, we must try

$\underline{y_p(x) = bx^2e^x}$ (multiplying e^x by lowest integer power of x, such that the result doesn't appear in $y_c(x)$)

Substituting this into the differential eq^n, we get

$$y_p(x) = bx^2e^x$$

$$\Rightarrow \frac{dy_p}{dx} = bx^2e^x + 2bxe^x$$

$$\Rightarrow \frac{d^2y_p}{dx^2} = bx^2e^x + 2bxe^x + 2bxe^x + 2be^x$$

So, $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$

$$\Rightarrow \cancel{bx^2e^x} + \cancel{2bxe^x} + \cancel{2bxe^x} + 2be^x - \cancel{2bx^2e^x} - \cancel{4bxe^x} + \cancel{bx^2e^x} = e^x$$

$$\Rightarrow 2be^x = e^x$$

$$\Rightarrow b = \frac{1}{2}.$$

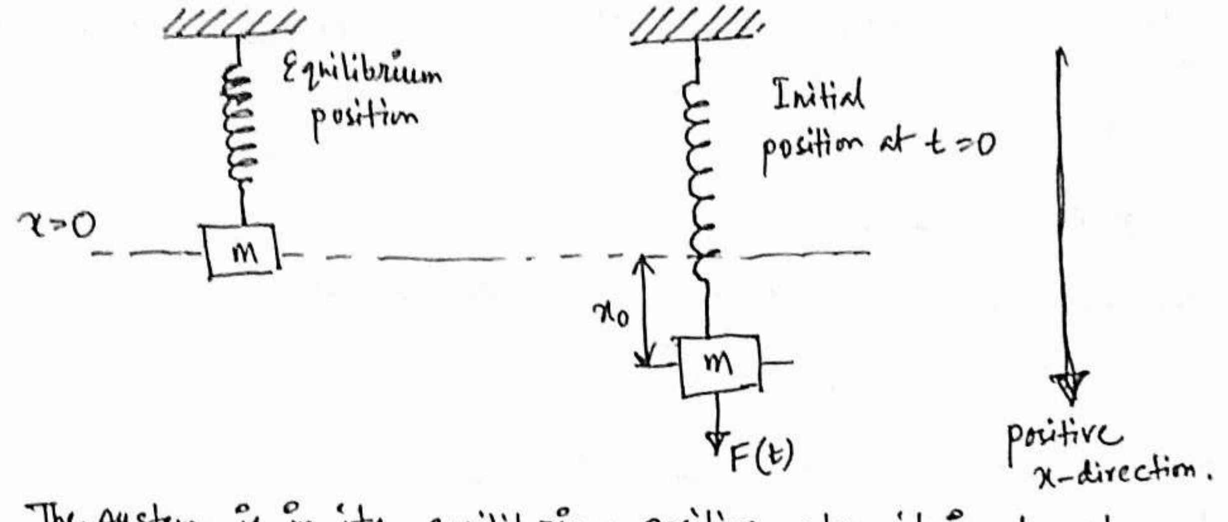
$$y_p(x) = \frac{1}{2}x^2e^x.$$

Thus the general sol^n becomes:

$$y(x) = y_c(x) + y_p(x) = \underline{(c_1 + c_2 x)e^x + \frac{1}{2}x^2e^x}.$$

(3) Applications of second-order Linear ODEs — Simple Harmonic Motion

Consider a simple spring system as shown in the figure below.



The system is in its <u>equilibrium position</u> when it is at rest. The mass is set in motion by one or more of the following means:

(a) displacing the mass from its equilibrium position
(b) providing it with an initial velocity
or, (c) applying an external force $F(t)$.

For convenience, we'll choose the downward direction as positive x-direction. We assume that, the mass of spring is negligible and the air resistance, if present, is proportional to the velocity of the body of mass m.

Thus, the most general case is that, at any time t, there are 3 forces acting on the system:

(a) a <u>restoring force</u> given by Hooke's law as

$$F_s = -kx, \quad k>0$$

F_s always acts in a direction that will tend to return the system to the equilibrium position: if the body is below the equilibrium position, then x is positive and F_s is negative; whereas if the body is above the equilibrium position, then x is negative and F_s is positive..

(b) a force due to air resistance given by

$$F_a = -a\dot{x}, \quad a>0 \qquad [\text{'}\cdot\text{' denotes } \tfrac{d}{dt}]$$

As a is positive, F_a always acts in the opposite direction of the velocity, and thus tends to <u>"retard"</u> or <u>"damp"</u> the motion of the body.

(c) an external force $F(t)$, measured in the positive direction.

From Newton's second law of motion, we can write

$$m\ddot{x} = F_{total} = -kx - a\dot{x} + F(t)$$

$$\Rightarrow \ddot{x} + \frac{a}{m}\dot{x} + \frac{k}{m}x = \frac{F(t)}{m} \quad \text{------} \quad (6)$$

If the system starts at $t=0$ with an initial velocity v_0 and from an initial position x_0, we then have the initial conditions:

$$x(0) = x_0 \quad \text{and} \quad \dot{x}(0) = v_0$$

Note that, the force of gravity doesn't explicitly appear in eq^n (6), ~~but~~ as we are measuring distance from the equilibrium position of the spring. If one wishes to exhibit gravity explicitly, then distance must be measured from the bottom-end of the "natural length" (i.e. length of the spring without anything attached to it, and neglecting the spring's mass). Then the motion will be given as:

$$\ddot{x} + \frac{a}{m}\dot{x} + \frac{k}{m}x = g + \frac{F(t)}{m}.$$

Equation (6) has the form of

$$\ddot{x} + a_1\dot{x} + a_0 x = f(t) \quad \text{------} \quad (7)$$

[2nd order linear ODE with constant coefficients.]

where $a_1 = \frac{a}{m}$, $a_0 = \frac{k}{m}$ and $f(t) = \frac{F(t)}{m}$.

In fact, simple LCR electrical circuits and ~~the~~ motion of floating bodies ~~as they~~ (i.e. buoyancy problems) – they all are governed by second-order linear differential eq^n of the form of eq^n (7), with ~~some~~ $a_1 = 0$ or $a_0 = 0$ or $f(t) = 0$. Note that, a_1 and a_0 are constants.

The motion of the spring system is classified as the following manner:

(a) Free and undamped: when $f(t) = 0$ (i.e. homogeneous ODE) and $a_1 = 0$

$$\ddot{x}(t) + a_0 x(t) = 0$$

(b) Free and damped: when $f(t) = 0$, but $a_1 \neq 0$.

$$\ddot{x}(t) + a_1\dot{x}(t) + a_0 x(t) = 0 \quad \text{(homogeneous ODE)}$$

For damped motion, there are 3 separate cases to consider, ~~as~~ according to the roots of the associated auxiliary equation:—

(i) roots are real and distinct — the motion is called "overdamped".

(ii) roots are equal — the motion is called "critically damped"

(iii) roots are complex conjugate — the motion is called "oscillatory damped" or sometimes "under damped."

(c) Forced : when $F(t) \neq 0$

$$\ddot{x}(t) + a_1 \dot{x}(t) + a_0 x(t) = F(t) \quad [\text{nonhomogeneous ODE}].$$

- A motion is called "transient", if it dies out (i.e. goes to zero) at $t \to \infty$.
- A motion is called "steady-state", if it's not transient.

E.g., free damped system always yield transient motion, while forced damped system may yield both transient and steady-state motion.

3.1: Solution of free undamped motion :

The second order linear ODE for this motion will be

$$\ddot{x}(t) + a_0 x(t) = 0$$

This type of motion can be achieved only by displacing the body from its equilibrium position — only the restoring force from Hooke's law, F_s will act on the system.
This is also known as "simple harmonic motion".

The auxiliary eq^n reads as:

$$\lambda^2 + a_0 = 0$$

$$\Rightarrow \lambda = \pm i\sqrt{a_0}$$

The roots are complex conjugate, $\lambda_1 = i\sqrt{a_0}$ and $\lambda_2 = -i\sqrt{a_0}$

Now introduce another variable $\omega = \sqrt{a_0} = \sqrt{\frac{k}{m}}$, which is often referred to as "angular frequency", it is related to the natural frequency n and the time period T as :

$$n = \frac{\omega}{2\pi}$$

$$\text{and} \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}.$$

As the roots $\lambda_1 = i\omega$ and $\lambda_2 = -i\omega$ are complex conjugate, the solution will be of the form

$$y(x) = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x) \quad \text{for } \lambda_1 = \alpha + i\beta,\ \lambda_2 = \alpha - i\beta.$$

So, in this case,

~~x(t) y(x) = C₁ cos~~

$$\boxed{x(t) = C_1 \cos \omega t + C_2 \sin \omega t}$$

The solution has an alternative form of

$$\boxed{x(t) = A \cos(\omega t - \phi)}$$

where the "amplitude" $A = \sqrt{C_1^2 + C_2^2}$ and the "phase angle" $\phi = \tan^{-1}\left(\frac{C_2}{C_1}\right)$.

3.2: Solution of free damped motion:

The second order linear ODE for this will be

$$\ddot{x}(t) + a_1 \dot{x}(t) + a_0 x(t) = 0$$

This type of motion is considered when force due to air resistance $F_a = -a\dot{x}$ is also present, and there are no external forces acting on the system $(F(t) = 0)$.

The auxiliary eqn reads as:

$$\lambda^2 + a_1 \lambda + a_0 = 0$$

$$\Rightarrow \lambda = \frac{-a_1 \pm \sqrt{a_1^2 - 4\omega^2}}{2} \quad [\text{where } \omega^2 = a_0]$$

Let, $D = a_1^2 - 4\omega^2$.

There are 3 cases to consider depending on the sign of D:

(i) If $\underset{\text{i.e. } a_1 > 2\omega}{D > 0}$, $\lambda_1 = \frac{-a_1 + \sqrt{D}}{2}$, $\lambda_2 = \frac{-a_1 - \sqrt{D}}{2}$

all the roots are real and distinct — the motion is called "overdamped".

The solution will be:

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

$$\Rightarrow \boxed{x(t) = C_1 \exp\left(\frac{-a_1 + \sqrt{D}}{2}\right) + C_2 \exp\left(\frac{-a_1 - \sqrt{D}}{2}\right)}$$

If the initial conditions are given as:

$$x(0) = x_0 \quad \text{and} \quad \dot{x}(0) = v_0$$

We can determine the values of the constants c_1, c_2 in terms of x_0 and v_0 by solving the following eq^{n}s:

$$c_1 + c_2 = x_0$$

and

$$c_1 \lambda_1 + c_2 \lambda_2 = v_0$$

$$\Rightarrow \quad c_1 = \frac{x_0 \lambda_2 - v_0}{\lambda_2 - \lambda_1}$$

$$c_2 = x_0 - \frac{x_0 \lambda_2 - v_0}{\lambda_2 - \lambda_1}$$

(ii) If $D = 0$; $\quad \lambda_1 = \lambda_2 = -\frac{a_1}{2}$.

and $D = a_1^2 - 4\omega^2 = 0 \Rightarrow a_1 = 2\omega$.

So, $\lambda_1 = \lambda_2 = -\omega$

As the roots are equal, the solution will be

$$\boxed{x(t) = (c_1 + c_2 t)\, e^{-\omega t}}$$

Such motion is called "critically damped".

Again the constants c_1 and c_2 can be determined from the initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$

$$c_1 = x_0$$

and

$$c_2 = v_0 + x_0 \omega$$

$$\dot{x}(t) = (c_1 + c_2 t)(-\omega) e^{-\omega t} + c_2 e^{-\omega t}$$

$$\dot{x}(0) = -c_1 \omega + c_2$$

$$\Rightarrow c_2 = \dot{x}(0) + c_1 \omega$$

(iii) If $D < 0$, i.e. $a_1 < 2\omega$, $\quad \lambda_1 = \frac{-a_1 + i\sqrt{D}}{2}$, $\lambda_2 = \frac{-a_1 - i\sqrt{D}}{2}$

introducing another variable,

$$\gamma = \frac{\sqrt{D}}{2},$$

$$\lambda_1 = -\frac{a_1}{2} + i\gamma, \quad \lambda_2 = -\frac{a_1}{2} - i\gamma.$$

As the roots are complex conjugate, the solution will be

$$\boxed{x(t) = e^{-\left(\frac{a_1}{2}\right)t} \left[c_1 \cos(\gamma t) + c_2 \sin(\gamma t)\right]}$$

$\leftarrow$ Such motion is called "underdamped" or "oscillatory damped".

c_1 and c_2 can be obtained from the initial conditions;

$$x(0) = x_0 \quad \text{and} \quad \dot{x}(0) = v_0$$

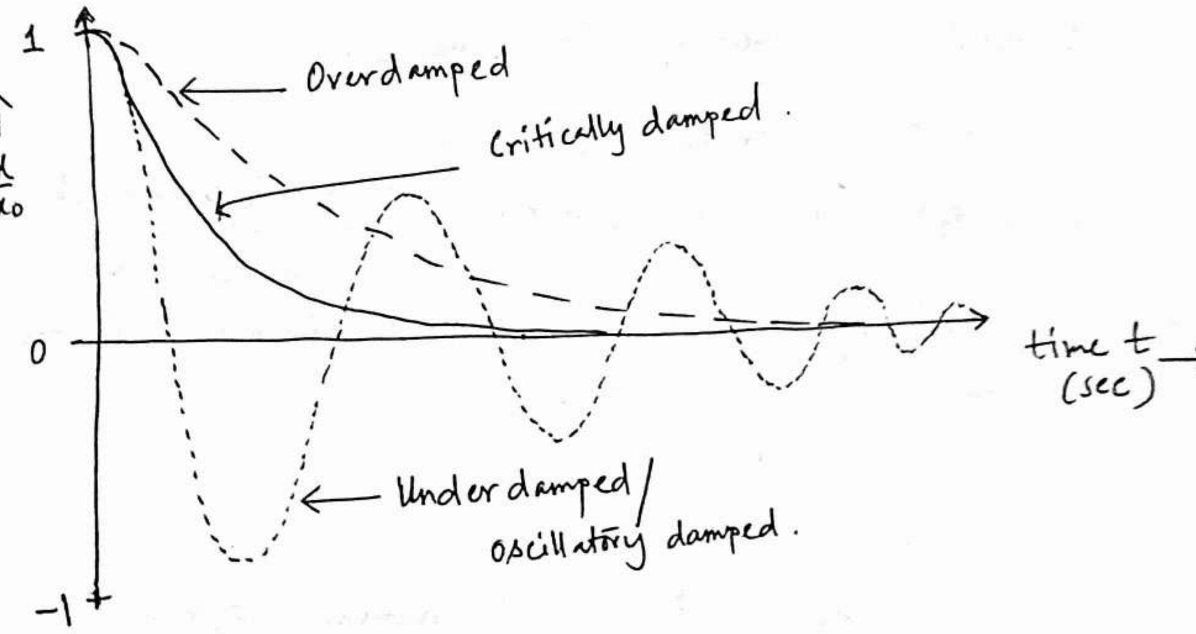
$$c_1 = x_0 \qquad \dot{x}(t) = -\left(\frac{a_1}{2}\right) e^{-\frac{a_1}{2}t}\left[c_1 \cos(\gamma t) + c_2 \sin(\gamma t)\right]$$

$$+ e^{-\frac{a_1}{2}t}\left[-\gamma c_1 \sin(\gamma t) + \gamma c_2 \cos(\gamma t)\right]$$

$$\dot{x}(0) = -\frac{a_1}{2} c_1 + \gamma c_2$$

$$\Rightarrow \gamma c_2 = v_0 + \frac{a_1}{2} x_0$$

$$\Rightarrow c_2 = \frac{v_0}{\gamma} + \frac{a_1 x_0}{2\gamma}.$$



3.3. : Solution of forced harmonic motion:

Differential eq^n for forced harmonic motion is a non-homogeneous ODE :

$$\ddot{x}(t) + a_1\dot{x}(t) + a_0 x(t) = F(t)$$

One can find the complementary function $y_c(x)$ using the above methods. Here also ~~various~~ two possible scenarios can be considered — undamped ($a_1 = 0$) and damped ($a_1 \neq 0$). We can study each cases for the given functional form of $F(t)$. Most popular choices of $F(t)$ are:

(i) Constant forcing — $F(t)$ = Constant, say C.

$$\ddot{x}(t) + a_1\dot{x}(t) + a_0 x(t) = C$$

and (ii) Sinusoidal forcing — $f(t) = \cancel{F_0 \cos(\gamma t)}\ f_0 \cos(w_e t)$ where w_e is some angular frequency, which is not necessarily related to w, and f_0 is a constant.

$$\ddot{x}(t) + a_1 \dot{x}(t) + a_0 x(t) = f_0 \cos(w_e t)$$

For this case, the particular integral $x_p(t)$ we are looking for has a form of:

$$\boxed{x_p(t) = b_1 \cos(w_e t) + b_2 \sin(w_e t)}$$

For instance, if we consider the forced undamped harmonic motion, i.e. $a_1 = 0$, we have

$$\ddot{x}(t) + w^2 x(t) = f_0 \cos(w_e t) \quad [\text{as } a_0 = w^2]$$

Substituting the ~~value of~~ $x_p(t)$, we get

$$\ddot{x}_p(t) + w^2 x_p(t) = f_0 \cos(w_e t)$$

$$\Rightarrow -w_e^2 \left(b_1 \cos(w_e t) + b_2 \sin(w_e t)\right) + w^2 \left(b_1 \cos(w_e t) + b_2 \sin(w_e t)\right) = f_0 \cos(w_e t)$$

$$\Rightarrow (w^2 - w_e^2) b_1 \cos(w_e t) + (w^2 - w_e^2) b_2 \sin(w_e t) = f_0 \cos(w_e t)$$

comparing the like-terms, we get,

$$b_2 = 0$$

$$(w^2 - w_e^2) b_1 = f_0$$

$$\Rightarrow b_1 = \frac{f_0}{w^2 - w_e^2} \quad [\text{assuming } w_e \neq w]$$

So, $$x_p(t) = \left(\frac{f_0}{w^2 - w_e^2}\right) \cdot \cos(w_e t)$$

We have already obtained the complementary function for undamped motion,

$$x_c(t) = c_1 \cos(wt) + c_2 \sin(wt)$$

Thus, the general solution will be —

$$x(t) = c_1 \cos(wt) + c_2 \sin(wt) + \left(\frac{f_0}{w^2 - w_e^2}\right) \cos(w_e t)$$

- If $w_e \neq w \longrightarrow$ this case is known as "Beats".
- If $w_e = w \longrightarrow$ this case is known as "Resonance".

In the $w_e = w$ case, we need to take the particular integral as — $\boxed{x_p(t) = t(b_1 \cos w_e t + b_2 \sin w_e t)}$.