Chapter 6

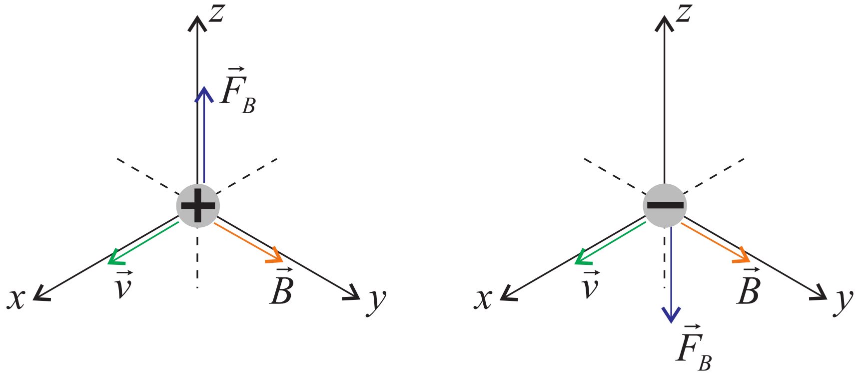
Magnetic Force

6.1 Magnetic Field

For stationary charges, they experienced an **electric force** in an **electric field**. For moving charges, they experienced a **magnetic force** in a **magnetic field**.

Mathematically, $\vec{F}_E = q\vec{E}$ (electric force)
$\vec{F}_B = q\vec{v} \times \vec{B}$ (magnetic force)

Direction of the magnetic force determined from *right hand rule*.



Magnetic field $\vec{B}$: Unit = Tesla (T)
1T = 1C moving at 1m/s experiencing 1N

Common Unit: 1 Gauss (G) = 10^{-4}T ≈ magnetic field on earth's surface

Example: What's the force on a 0.1C charge moving at velocity $\vec{v} = (10\hat{j} - 20\hat{k})ms^{-1}$ in a magnetic field $\vec{B} = (-3\hat{i} + 4\hat{k}) \times 10^{-4}T$

$$\vec{F} = q\vec{v} \times \vec{B}$$

$$\begin{aligned} &= +0.1\,(10\hat{j} - 20\hat{k}) \times (-3\hat{i} + 4\hat{k}) \times 10^{-4} N \\ &= 10^{-5}\,(-30 \cdot -\hat{k} + 40\hat{i} + 60\hat{j} + 0) N \end{aligned}$$

Effects of magnetic field is usually quite small.

$$\begin{aligned} \vec{F} &= q\vec{v} \times \vec{B} \\ |\vec{F}| &= qvB\sin\theta, \quad \text{where } \theta \text{ is the angle between } \vec{v} \text{ and } \vec{B} \end{aligned}$$

$\therefore$ Magnetic force is *maximum* when $\theta = 90°$ (i.e. $\vec{v} \perp \vec{B}$)
Magnetic force is *minimum* (0) when $\theta = 0°,\ 180°$ (i.e. $\vec{v} \parallel \vec{B}$)

Graphical representation of B-field: **Magnetic field lines**
Compared with **Electric field lines**:

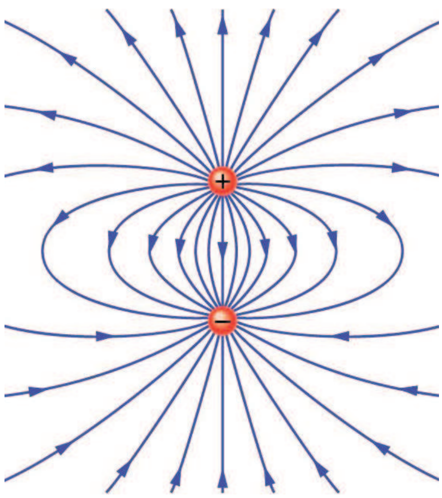
Similar characteristics :

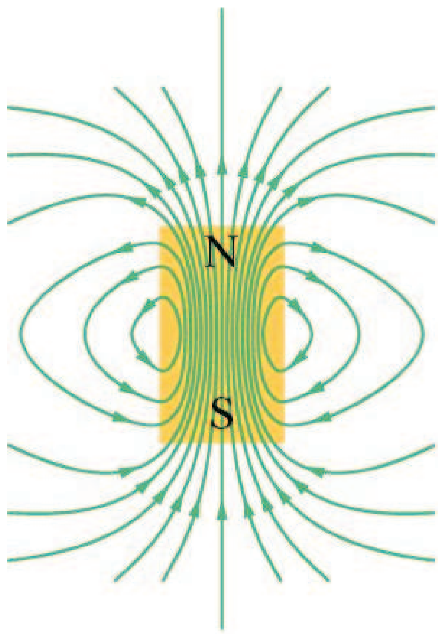
(1) Direction of E-field/B-field indicated by *tangent* of the field lines.
(2) Magnitude of E-field/B-field indicated by *density* of the field lines.

Differeces :

(1) $\vec{F}_E \parallel$ E-field lines; $\vec{F}_B \perp$ B-field lines
(2) E-field line begins at positive charge and ends at negative charge; B-field line forms a closed loop.

Example : Chap35, Pg803 Halliday



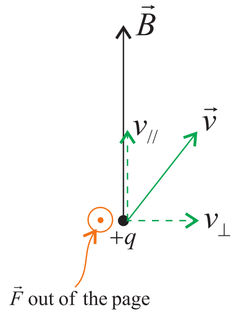


Note: Isolated magnetic monopoles do not exist.

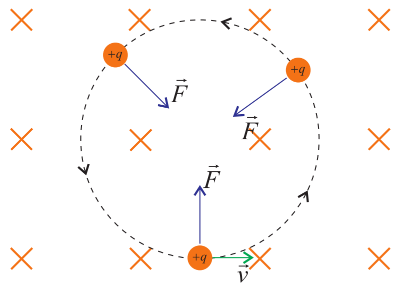
6.2 Motion of A Point Charge in Magnetic Field

Since $\vec{F}_B \perp \vec{v}$, therefore B-field only changes the *direction* of the velocity but not its *magnitude*.

Generally, $\vec{F}_B = q\vec{v} \times \vec{B} = q\, v_\perp B$,
$\therefore$ We only need to consider the motion component $\perp$ to B-field.



We have *circular motion*. Magnetic force provides the *centripetal force* on the moving charge particles.



$$\begin{aligned} \therefore \quad F_B &= m\,\frac{v^2}{r} \\ |q|\, vB &= m\,\frac{v^2}{r} \\ \therefore \quad r &= \frac{mv}{|q|B} \end{aligned}$$

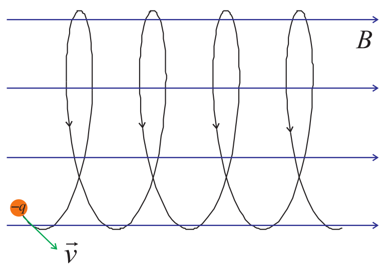
where r is radius of circular motion.

Time for moving around one orbit:

$$\boxed{T = \frac{2\pi r}{v} = \frac{2\pi m}{qB}} \quad \text{Cyclotron Period}$$

(1) Independent of v (non-relativistic)

(2) Use it to measure m/q



Generally, charged particles with constant velocity moves in **helix** in the presence of constant B-field.

Note :

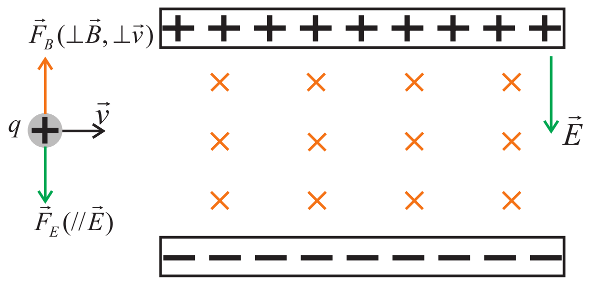
(1) B-field does NO work on particles.

(2) B-field does NOT change K.E. of particles.

Particle Motion in Presence of E-field & B-field:

$$\boxed{\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}} \qquad \text{Lorentz Force}$$

Special Case : $\vec{E} \perp \vec{B}$



$$\text{When} \quad |\vec{F}_E| = |\vec{F}_B|$$
$$qE = qvB$$
$$\Rightarrow \quad \boxed{v = \frac{E}{B}}$$

∴ For charged particles moving at $v = E/B$, they will pass through the *crossed E and B fields* without vertical displacement.

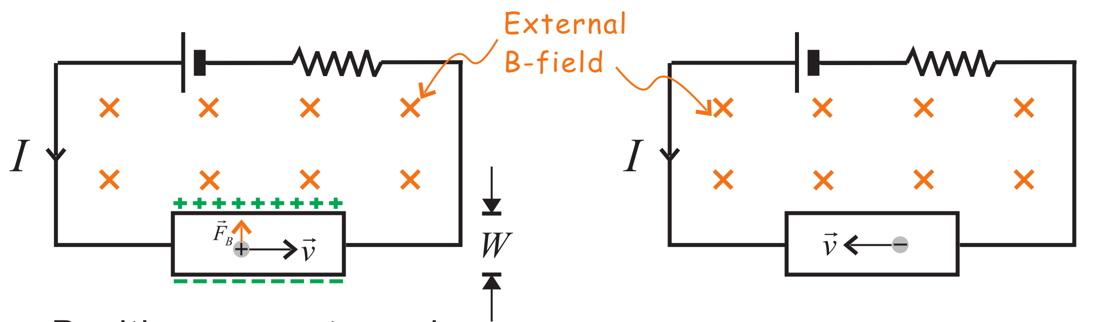
⇒ **velocity selector**

Applications :

- Cyclotron (Lawrence & Livingston 1934)
- Measuring e/m for electrons (Thomson 1897)
- Mass Spectrometer (Aston 1919)

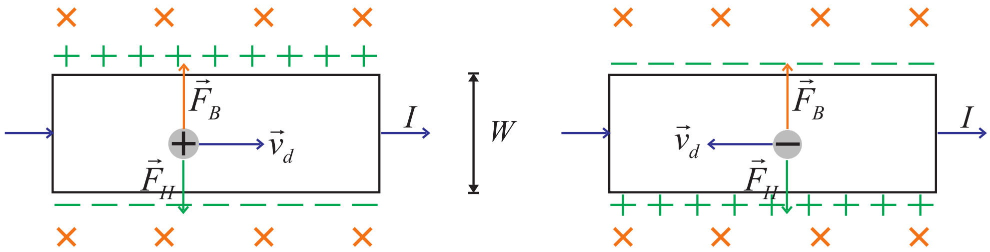
6.3 Hall Effect

Charges travelling in a conducting wire will be *pushed to one side of the wire by the external magnetic field*. This separation of charge in the wire is called the **Hall Effect**.



Positive current carrier   Negative current carrier

The separation will stop when F_B experienced by the current carrier is *balanced* by the force $\vec{F}_H$ caused by the E-field set up by the separated charges.



Define :

$$\Delta V_H = \textbf{Hall Voltage}$$
$$= \text{Potential difference across the conducting strip}$$

$\therefore$ E-field from separated charges: $E_H = \dfrac{\Delta V_H}{W}$
where W = *width* of conducting strip

In equilibrium: $q\vec{E}_H + q\vec{v}_d \times \vec{B} = 0$, where $\vec{v}_d$ is drift velocity

$$\therefore \quad \frac{\Delta V_H}{W} = v_d B$$

Recall from Chapter 5,

$$i = nqAv_d$$

where n is density of charge carrier,
A is cross-sectional area = width × thickness = $W \cdot t$

$$\therefore \quad \frac{\Delta V_H}{W} = \frac{i}{nqWt} B$$

$$\Rightarrow \quad \boxed{n = \frac{iB}{qt\Delta V_H}}$$

To determine density of charge carriers

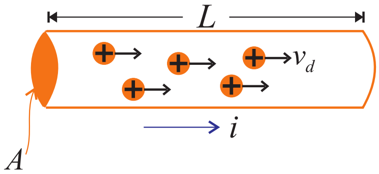
Suppose we determine n for a particular metal ($\therefore\ q = e$), then we can *measure B-field strength by measuring the Hall voltage*:

$$\boxed{B = \frac{net}{i} \Delta V_H}$$

6.4 Magnetic Force on Currents

Current = many charges moving together

Consider a wire segment, length L, carrying current i in a magnetic field.



Total magnetic force = $\Big(\underbrace{q\vec{v}_d \times \vec{B}}_{\text{force on one charge carrier}}\Big) \cdot \underbrace{n\,A\,L}_{\text{Total number of charge carrier}}$

Recall $i = nqv_dA$

$$\therefore \quad \boxed{\text{Magnetic force on current } \vec{F} = i\vec{L} \times \vec{B}}$$

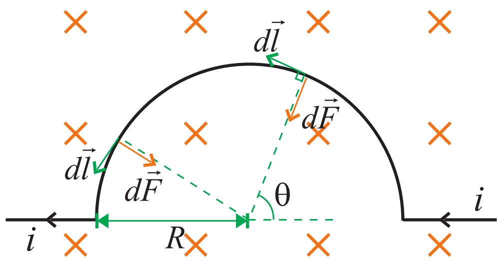
where $\vec{L}$ = Vector of which: $|\vec{L}|$ = length of current segment; direction = direction of current

For an infinitesimal wire segment $d\vec{l}$

$$\boxed{d\vec{F} = i\,d\vec{l} \times \vec{B}}$$

Example 1: Force on a semicircle current loop

$$\begin{aligned} d\vec{l} &= \text{Infinitesimal} \\ &\quad\ \text{arc length element} \perp \vec{B} \\ \therefore \quad dl &= R\,d\theta \\ \therefore \quad dF &= iRB\,d\theta \end{aligned}$$



By symmetry argument, we only need to consider vertical forces, $dF \cdot \sin\theta$

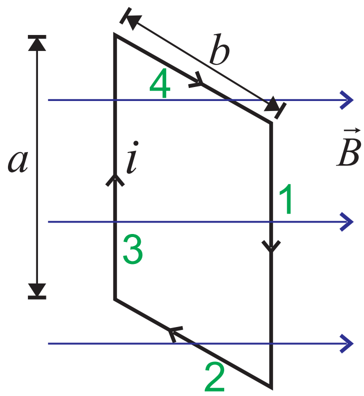
$$\begin{aligned} \therefore \quad \text{Net force } F &= \int_0^{\pi} dF\ \sin\theta \\ &= iRB \int_0^{\pi} \sin\theta\,d\theta \\ F &= 2iRB \quad \text{(downward)} \end{aligned}$$

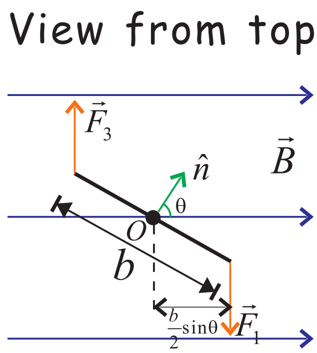
Method 2: Write $d\vec{l}$ in $\hat{\imath}, \hat{\jmath}$ components

$$
\begin{aligned}
d\vec{l} &= -dl\sin\theta\,\hat{\imath} + dl\cos\theta\,\hat{\jmath} \\
&= R\,d\theta\,(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}) \\
\vec{B} &= -B\,\hat{k} \quad \text{(into the page)} \\
\therefore \quad d\vec{F} &= i\,d\vec{l} \times \vec{B} \\
&= -iRB\sin\theta\,d\theta\,\hat{\jmath} - iRB\cos\theta\,\hat{\imath}
\end{aligned}
$$

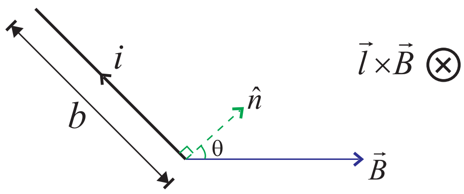
$$
\begin{aligned}
\therefore \quad \vec{F} &= \int_0^{\pi} d\vec{F} \\
&= -iRB\left[\int_0^{\pi}\sin\theta\,d\theta\,\hat{\jmath} + \int_0^{\pi}\cos\theta\,d\theta\,\hat{\imath}\right] \\
&= -2iRB\,\hat{\jmath}
\end{aligned}
$$

Example 2: Current loop in B-field



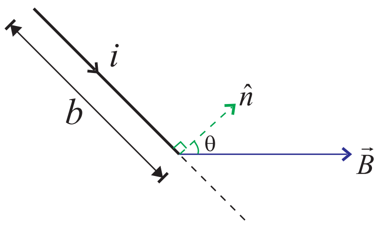


For segment2:



$$F_2 = ibB\sin(90^\circ + \theta) = ibB\cos\theta \qquad \text{(pointing downward)}$$

For segment4:



$$F_2 = ibB\sin(90^\circ - \theta) = ibB\cos\theta \qquad \text{(pointing upward)}$$

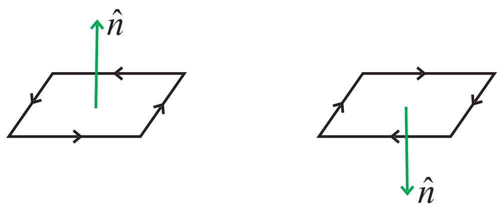
For segment1: $F_1 = iaB$
For segment3: $F_3 = iaB$
$\therefore$ Net force on the current loop $= 0$
But, net torque on the loop about O

$$\begin{aligned} &= \tau_1 + \tau_3 \\ &= iaB \cdot \frac{b}{2} \sin\theta + iaB \cdot \frac{b}{2} \sin\theta \\ &= i \underbrace{ab}_{A = \text{area of loop}} B \sin\theta \end{aligned}$$

Suppose the loop is a coil with N turns of wires:

$$\text{Total torque } \boxed{\tau = NiAB \sin\theta}$$

Define: Unit vector $\hat{n}$ to represent the area-vector (using right hand rule)



Then we can rewrite the torque equation as

$$\boxed{\vec{\tau} = NiA\,\hat{n} \times \vec{B}}$$

Define: $NiA\,\hat{n} = \vec{\mu} =$ Magnetic dipole moment of loop

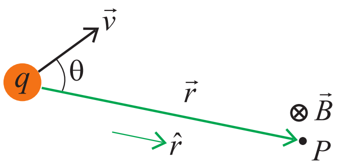
$$\boxed{\vec{\tau} = \vec{\mu} \times \vec{B}}$$

Chapter 7

Magnetic Field

7.1 Magnetic Field

A *moving charge* $\begin{cases} \text{experiences magnetic force in B-field.} \\ \text{can generate B-field.} \end{cases}$



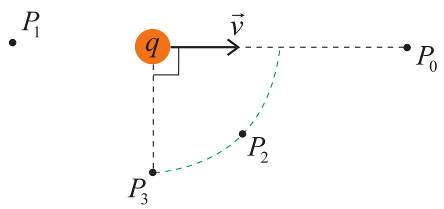
Magnetic field $\vec{B}$ due to moving point charge:

$$\vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{q\vec{v} \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi} \cdot \frac{q\vec{v} \times \vec{r}}{r^3}$$

where $\mu_0 = 4\pi \times 10^{-7}$ Tm/A (N/A^2)

Permeability of free space (Magnetic constant)

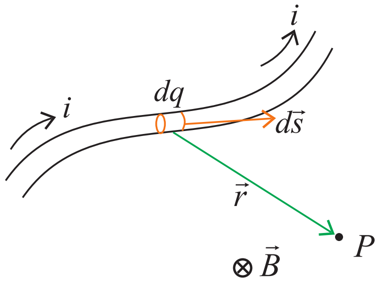
$$|\vec{B}| = \frac{\mu_0}{4\pi} \cdot \frac{qv\sin\theta}{r^2} \quad \begin{cases} \textit{maximum} & \text{when } \theta = 90^\circ \\ \textit{minimum} & \text{when } \theta = 0^\circ/180^\circ \end{cases}$$



$\vec{B}$ at $P_0 = 0 = \vec{B}$ at P_1
$\vec{B}$ at $P_2 < \vec{B}$ at P_3

However, a single moving charge will NOT generate a steady magnetic field.

stationary charges	generate	steady E-field.
steady currents	generate	steady B-field.



Magnetic field at point P can be obtained by *integrating* the contribution from individual current segments. (**Principle of Superposition**)

$$\therefore \quad d\vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{dq\,\vec{v} \times \hat{r}}{r^2}$$

Notice: $\quad dq\,\vec{v} = dq \cdot \dfrac{d\vec{s}}{dt} = i\,d\vec{s}$

$$\boxed{d\vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{i\,d\vec{s} \times \hat{r}}{r^2}} \qquad \textbf{Biot-Savart Law}$$

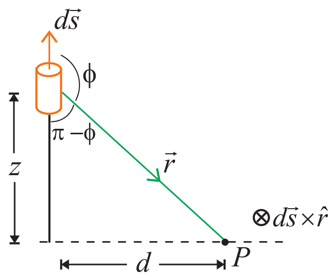
For current around a whole circuit:

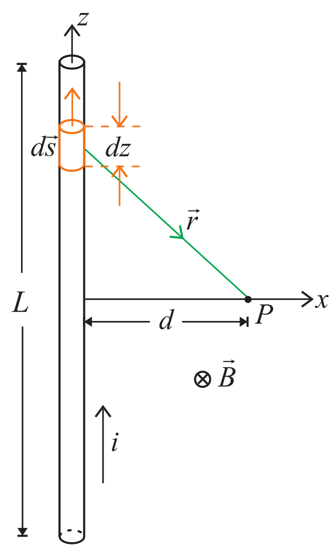
$$\vec{B} = \int_{\substack{\text{entire}\\\text{circuit}}} d\vec{B} = \int_{\substack{\text{entire}\\\text{circuit}}} \frac{\mu_0}{4\pi} \cdot \frac{i\,d\vec{s} \times \hat{r}}{r^2}$$

Biot-Savart Law is to *magnetic field* as
Coulomb's Law is to *electric field.*

Basic element of E-field: *Electric charges* dq
Basic element of B-field: *Current element* $i\,d\vec{s}$

Example 1 : Magnetic field due to straight current segment





$$
\begin{aligned}
\therefore\ |d\vec{s}\times\hat{r}| &= dz\sin\phi \\
&= dz\sin(\pi-\phi) \qquad \text{(Trigonometry Identity)} \\
&= dz\cdot\frac{d}{r} = \frac{d\cdot dz}{\sqrt{d^2+z^2}}
\end{aligned}
$$

$$dB = \frac{\mu_0}{4\pi}\cdot\frac{i\,dz}{r^2}\cdot\frac{d}{r} = \frac{\mu_0 i}{4\pi}\cdot\frac{d}{(d^2+z^2)^{3/2}}\,dz$$

$$\therefore\quad B = \int_{-L/2}^{L/2} dB = \frac{\mu_0 i d}{4\pi}\int_{-L/2}^{+L/2}\frac{dz}{(d^2+z^2)^{3/2}}$$

$$
\begin{aligned}
B &= \frac{\mu_0 i}{4\pi d}\cdot\frac{z}{(z^2+d^2)^{1/2}}\Bigg|_{-L/2}^{+L/2} \\
B &= \frac{\mu_0 i}{4\pi d}\cdot\frac{L}{(\frac{L^2}{4}+d^2)^{1/2}}
\end{aligned}
$$

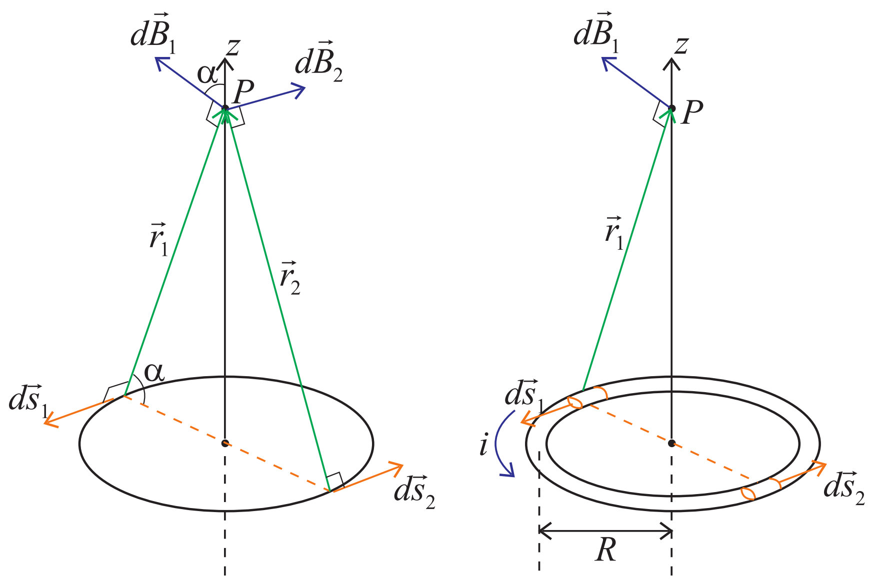
Limiting Cases : When $L \gg d$ (B-field due to long wire)

$$\Big(\frac{L^2}{4}+d^2\Big)^{-1/2} \approx \Big(\frac{L^2}{4}\Big)^{-1/2} = \frac{2}{L}$$

$$\therefore \quad \boxed{B = \frac{\mu_0 i}{2\pi d}\,;\quad \text{direction of B-field determined from } \textit{right-hand screw rule}}$$

Recall : $E = \dfrac{\lambda}{2\pi\epsilon_0 d}$ for an infinite long line of charge.

Example 2 : A circular current loop



Notice that for every current element $id\vec{s}_1$, generating a magnetic field $d\vec{B}_1$ at point P, there is an opposite current element $id\vec{s}_2$, generating B-field $d\vec{B}_2$ so that

$$d\vec{B}_1 \sin\alpha = -d\vec{B}_2 \sin\alpha$$

$\therefore$ Only vertical component of B-field needs to be considered at point P.

$$dB = \frac{\mu_0}{4\pi} \cdot \frac{i\,ds\,\sin\overbrace{90^\circ}^{\because d\vec{s}\perp\hat{r}}}{r^2}$$

$\therefore$ B-field at point P:

$$B = \underbrace{\int}_{\text{around circuit}} dB \underbrace{\cos\alpha}_{\text{consider vertical component}}$$

$$\begin{aligned}
\therefore\quad B &= \int_0^{2\pi} \frac{\mu_0 i \cos\alpha}{4\pi r^2} \cdot \underbrace{ds}_{Rd\theta} \\
&= \frac{\mu_0 i}{4\pi} \cdot \frac{R}{r^3} \underbrace{\int_0^{2\pi} ds}_{\text{Integrate around circumference of circle } = 2\pi R} \\
\therefore\quad B &= \frac{\mu_0 i R^2}{2r^3}
\end{aligned}$$

$$\boxed{B = \frac{\mu_0 i R^2}{2(R^2+z^2)^{3/2}}\ ; \quad \text{direction of B-field determined from \textit{right-hand screw rule}}}$$

Limiting Cases :

(1) B-field at center of loop:

$$z = 0 \quad \Rightarrow \quad \boxed{B = \frac{\mu_0 i}{2R}}$$

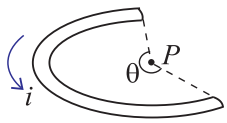
(2) For $z \gg R$,

$$B = \frac{\mu_0 i R^2}{2z^3\left(1+\frac{R^2}{z^2}\right)^{3/2}} \approx \frac{\mu_0 i R^2}{2z^3} \propto \frac{1}{z^3}$$

Recall E-field for an electric dipole: $E = \dfrac{p}{4\pi\epsilon_0 x^3}$

$\therefore$ A circular current loop is also called a **magnetic dipole**.

(3) A current arc:



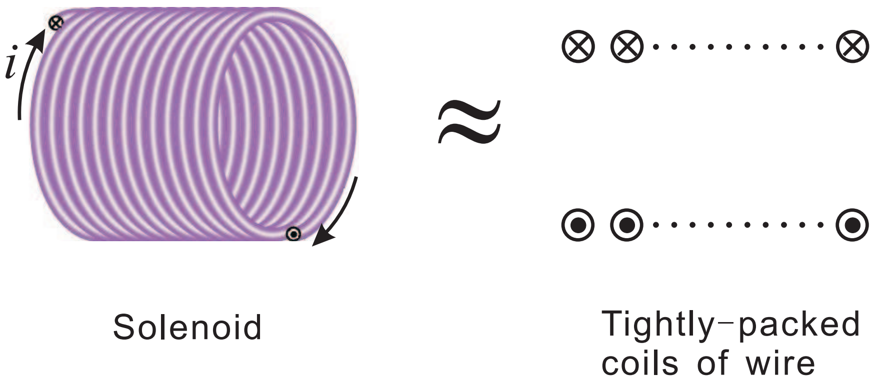
$$B = \int_{\substack{\text{around}\\ \text{circuit}}} dB \underbrace{\cos\alpha}_{\substack{z=0 \;\Rightarrow\\ \alpha = 0 \;\text{ here.}}}$$

$$= \frac{\mu_0 i}{4\pi}\cdot \underbrace{\frac{R}{r^3}}_{\substack{R=r\\ \text{when } \alpha=0}} \cdot \overbrace{\int_0^\theta \underbrace{ds}_{Rd\theta}}^{R\theta = \text{length of arc}}$$

$$B = \frac{\mu_0 i\,\theta}{4\pi R}$$

Example 3 : Magnetic field of a solenoid

Solenoid is used to produce a *strong and uniform* magnetic field inside its coils.



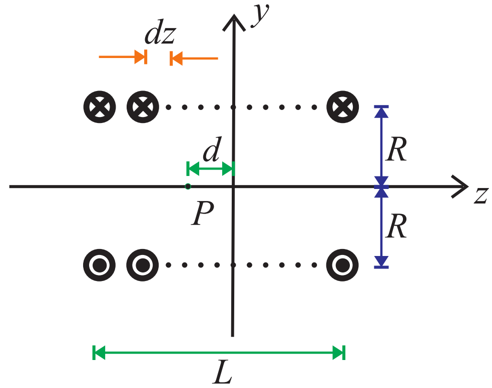
Consider a solenoid of length L consisting of N turns of wire.

Define: n = Number of turns per unit length $= \dfrac{N}{L}$

Consider B-field at distance d from the center of the solenoid:

For a segment of length dz, number of current turns $= ndz$

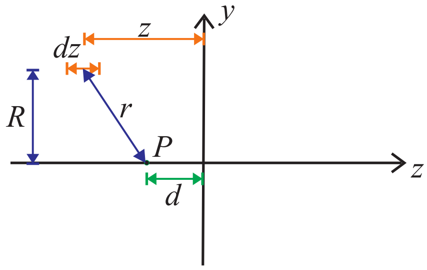
$\therefore$ Total current $= ni\,dz$



Using the result from one coil in Example 2, we get B-field from coils of length dz at distance z from center:

$$dB = \frac{\mu_0 (ni\, dz) R^2}{2r^3}$$

However $r = \sqrt{R^2 + (z-d)^2}$



$$\begin{aligned} \therefore \quad B &= \int_{-L/2}^{+L/2} dB \quad \text{(Integrating over the entire solenoid)} \\ &= \frac{\mu_0 ni R^2}{2} \int_{-L/2}^{+L/2} \frac{dz}{[R^2 + (z-d)^2]^{3/2}} \\ B &= \frac{\mu_0 ni}{2} \left[\frac{\frac{L}{2} + d}{\sqrt{R^2 + (\frac{L}{2} + d)^2}} + \frac{\frac{L}{2} - d}{\sqrt{R^2 + (\frac{L}{2} - d)^2}} \right] \end{aligned}$$

along negative z direction

Ideal Solenoid :

$$L \gg R$$

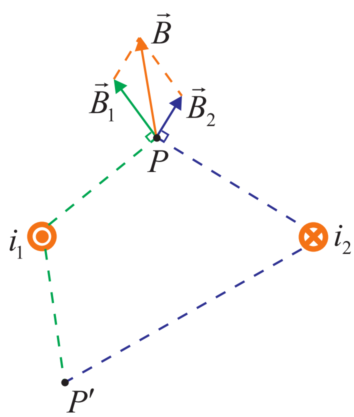
$$\text{then} \quad B = \frac{\mu_0 ni}{2} [1 + 1]$$

$$\therefore \quad \boxed{B = \mu_0 ni\,; \quad \text{direction of B-field determined from } \textit{right-hand screw rule}}$$

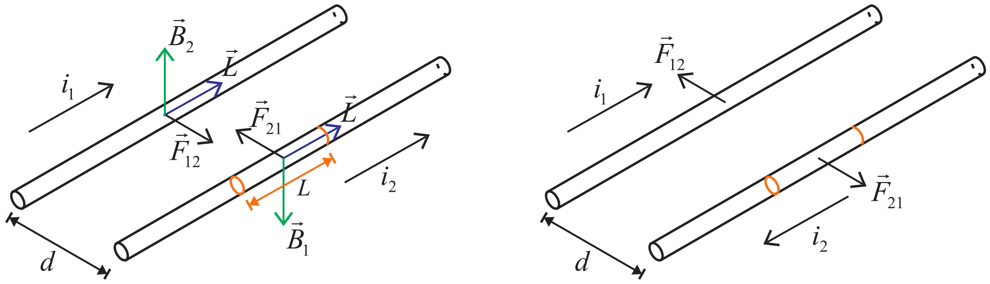
Question : What is the B-field at the end of an ideal solenoid? $B = \frac{\mu_0 ni}{2}$

7.2 Parallel Currents

Magnetic field at point P $\vec{B}$ due to two currents i_1 and i_2 is the *vector sum* of the $\vec{B}$ fields $\vec{B}_1$, $\vec{B}_2$ due to individual currents. **(Principle of Superposition)**



Force Between Parallel Currents :



Consider a segment of length L on i_2 :

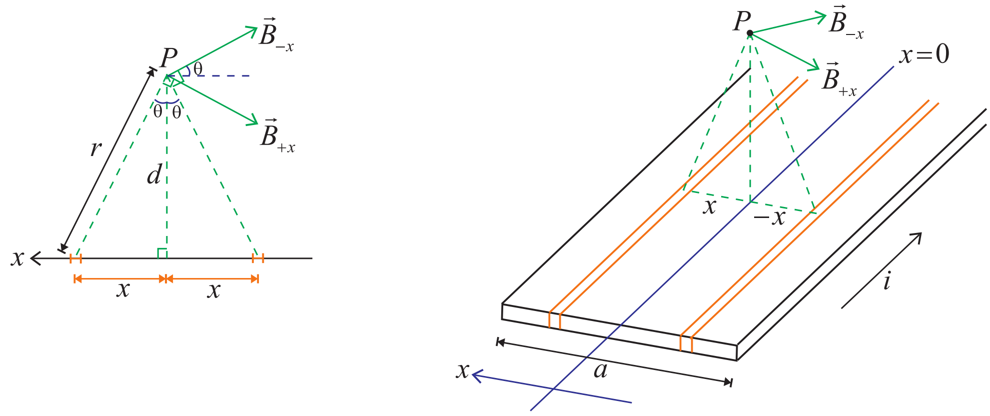
$$\vec{B}_1 = \frac{\mu_0 i_1}{2\pi d} \quad \text{(pointing down)} \qquad\qquad \vec{B}_2 = \frac{\mu_0 i_1}{2\pi d} \quad \text{(pointing up)}$$

Force on i_2 coming from i_1:

$$|\vec{F}_{21}| = i_2\vec{L} \times \vec{B}_1 = \frac{\mu_0 L i_1 i_2}{2\pi d} = |\vec{F}_{12}| \qquad \textit{(Def'n of ampere, A)}$$

$\therefore$ *Parallel currents attract, anti-parallel currents repel.*

Example : Sheet of current



Consider an infinitesimal wire of width dx at position x, there exists another element at $-x$ so that vertical $\vec{B}$-field components of $\vec{B}_{+x}$ and $\vec{B}_{-x}$ cancel.

$\therefore$ Magnetic field due to dx wire:

$$dB = \frac{\mu_0 \cdot di}{2\pi r} \qquad \text{where } di = i\Big(\frac{dx}{a}\Big)$$

$\therefore$ Total B-field (*pointing along* $-x$ *axis*) at point P:

$$B = \int_{-a/2}^{+a/2} dB\cos\theta = \int_{-a/2}^{+a/2} \frac{\mu_0 i}{2\pi a} \cdot \frac{dx}{r} \cdot \cos\theta$$

Variable transformation (Goal: change r, x to d, θ, then integrate over θ):

$$\begin{cases} d = r\cos\theta & \Rightarrow \quad r = d\sec\theta \\ x = d\tan\theta & \Rightarrow \quad dx = d\sec^2\theta\, d\theta \end{cases}$$

Limits of integration: $-\theta_0$ to θ_0, where $\tan\theta_0 = \dfrac{a}{2d}$

$$\begin{aligned} \therefore \quad B &= \frac{\mu_0 i}{2\pi a}\int_{-\theta_0}^{\theta_0} \frac{d\sec^2\theta\, d\theta}{d\sec\theta}\cdot\cos\theta \\ &= \frac{\mu_0 i}{2\pi a}\int_{-\theta_0}^{\theta_0} d\theta \\ B &= \frac{\mu_0 i\theta_0}{\pi a} = \frac{\mu_0 i}{\pi a}\tan^{-1}\Big(\frac{a}{2d}\Big) \end{aligned}$$

Limiting Cases :

(1) $d \gg a$

$$\tan\theta = \frac{a}{2d} \quad \Rightarrow \quad \theta \approx \frac{a}{2d}$$

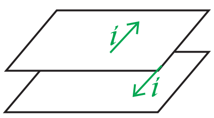
$$\therefore \quad B = \frac{\mu_0 i}{2\pi a}$$ B-field due to infinite long wire

(2) $d \ll a$ (*Infinite sheet of current*)

$$\tan\theta = \frac{a}{2d} \to \infty \quad \Rightarrow \quad \theta = \frac{\pi}{2}$$

$$\therefore \quad B = \frac{\mu_0 i}{2a} \qquad \textit{Constant}!$$

Question : Large sheet of opposite flowing currents.



What's the B-field between & outside the sheets?

7.3 Ampère's Law

In our study of electricity, we notice that the **inverse square force law** leads to **Gauss' Law**, which is useful for *finding E-field for systems with high level of symmetry.*
For magnetism, Gauss' Law is simple

$$\oiint_S \qquad \boxed{\oint_S \vec{B} \cdot d\vec{A} = 0}$$

$\because$ There is no magnetic monopole.

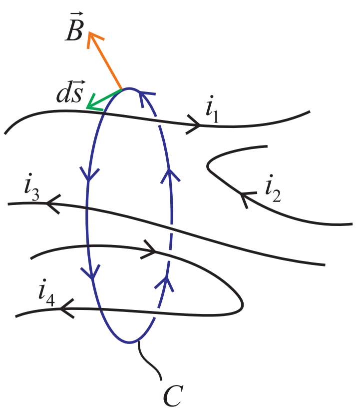
A more useful law for calculating B-field for highly symmetric situations is **the Ampère's Law**:

$$\oint_C \qquad \boxed{\oint_C \vec{B} \cdot d\vec{s} = \mu_0 i}$$

$\oint_C$ = *Line intefral* evaluated around a *closed loop C* **(Amperian curve)**

i = *Net current* that penetrates the area bounded by curve C^* (topological property)

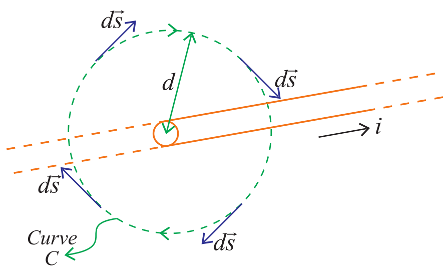
Convention : Use the **right-hand screw rule** to determine the *sign* of current.



$$\begin{aligned}\oint_C \vec{B} \cdot d\vec{s} &= \mu_0(i_1 - i_3 + i_4 - i_4) \\ &= \mu_0(i_1 - i_3)\end{aligned}$$

Applications of the Ampere's Law :

(1) Long-straight wire



Construct an Amperian curve of radius d:

By symmetry argument, we know $\vec{B}$-field only has *tangential component*

$$\therefore \quad \oint_C \vec{B} \cdot d\vec{s} = \mu_0 i$$

Take $d\vec{s}$ to be the tangential vector around the circular path:

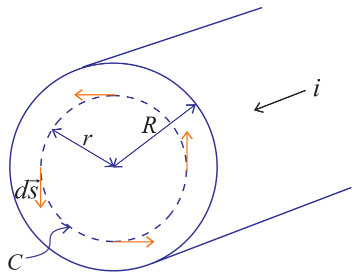
$$\begin{aligned}
\therefore \quad \vec{B}\cdot d\vec{s} &= B\,ds \\
B \underbrace{\oint_C ds}_{\substack{\text{Circumference} \\ \text{of circle} = 2\pi d}} &= \mu_0 i \\
\therefore \quad B(2\pi d) &= \mu_0 i
\end{aligned}$$

B-field due to long, straight current

$$\boxed{B = \frac{\mu_0 i}{2\pi d}}$$

(Compare with 7.1 Example 1)

(2) Inside a current-carrying wire

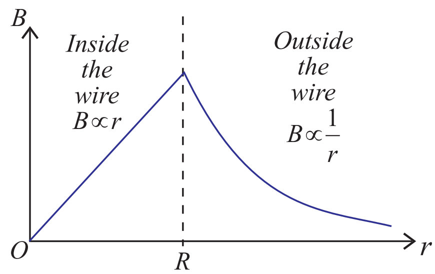


Again, symmetry argument implies that $\vec{B}$ is *tangential* to the Amperian curve and $\vec{B} \to B(r)\hat{\theta}$

Consider an Amperian curve of radius $r(< R)$

$$\oint_C \vec{B}\cdot d\vec{s} = B\oint ds = B(2\pi r) = \mu_0 i_{included}$$

But $i_{included} \propto$ cross-sectional area of C

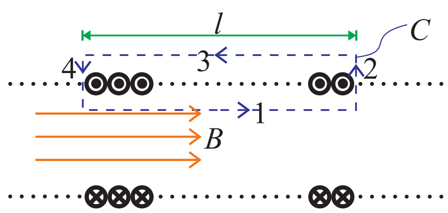


$$\begin{aligned}
\therefore \quad \frac{i_{included}}{i} &= \frac{\pi r^2}{\pi R^2} \\
\therefore \quad i_{included} &= \frac{r^2}{R^2}\, i \\
\therefore \quad B &= \frac{\mu_0 i}{2\pi R^2}\cdot r \propto r
\end{aligned}$$

Recall: Uniformly charged infinite long rod

(3) Solenoid (Ideal)

Consider the rectangular Amperian curve 1234.



$$\oint_C \vec{B}\cdot d\vec{s} = \int_1 \vec{B}\cdot d\vec{s} + \cancel{\int_2 \vec{B}\cdot d\vec{s}} + \cancel{\int_3 \vec{B}\cdot d\vec{s}} + \cancel{\int_4 \vec{B}\cdot d\vec{s}}$$

$$\int_2 = \int_4 = 0 \quad \because \quad \begin{cases} \vec{B}\cdot d\vec{s} &= 0 \quad \text{inside solenoid} \\ \vec{B} &= 0 \quad \text{outside solenoid} \end{cases}$$

$$\int_3 = 0 \quad \because \quad \vec{B} = 0 \quad \text{outside solenoid}$$

$$\therefore \quad \oint_C \vec{B}\cdot d\vec{s} = \int_1 \vec{B}\cdot d\vec{s} = Bl = \mu_0 i_{tot}$$

But $i_{tot} = \underbrace{nl}_{\text{Number of coils included}} \cdot i$

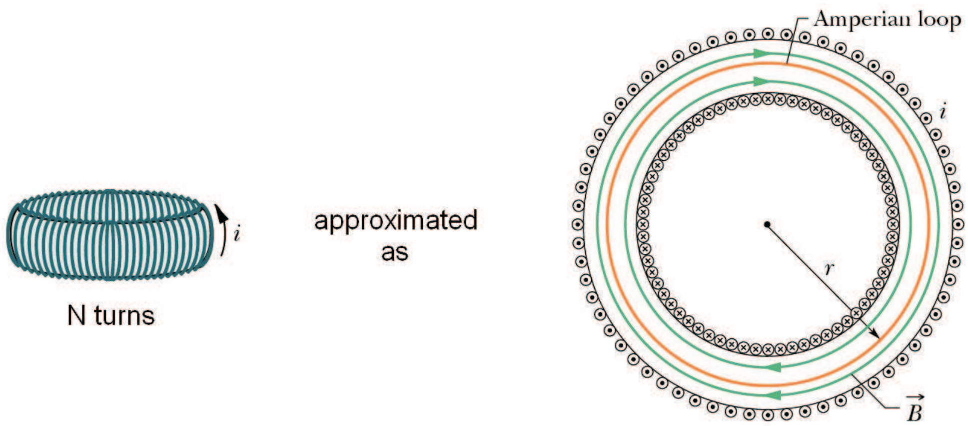
$$\therefore \boxed{B = \mu_0 n i}$$

Note :

(i) The assumption that $\vec{B} = 0$ outside the ideal solenoid is only *approximate*. (Halliday, Pg.763)

(ii) B-field everywhere inside the solenoid is a *constant*. (for ideal solenoid)

(4) Toroid (A *circular solenoid*)



By symmetry argument, the B-field lines form *concentric circles inside the toroid*.

Take Amperian curve C to be a circle of radius r inside the toroid.

$$\oint_C \vec{B}\cdot d\vec{s} = B\oint_C ds = B\cdot 2\pi r = \mu_0(Ni)$$

$$\therefore \quad B = \frac{\mu_0 N i}{2\pi r} \qquad \text{inside toroid}$$

Note :

(i) $B \neq$ constant inside toroid

(ii) Outside toroid:
Take Amperian curve to be circle of radius $r > R$.

$$\oint_C \vec{B} \cdot d\vec{s} = B \oint_C ds = B \cdot 2\pi r = \mu_0 \cdot i_{incl} = 0$$

$$\therefore \quad B = 0$$

Similarly, in the central cavity $B = 0$

7.4 Magnetic Dipole

Recall from §6.4, we define the **magnetic dipole moment** of a rectangular current loop

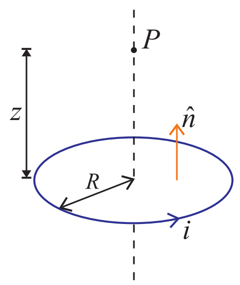
$$\boxed{\vec{\mu} = NiA\hat{n}}$$

where $\hat{n}$ = area unit vector with direction determined by the right-hand rule
N = Number of turns in current loop
A = Area of current loop

This is actually a *general definition* of a magnetic dipole, `i.e.` we use it for current loops of <u>all</u> shapes.

A common and symmetric example: circular current.

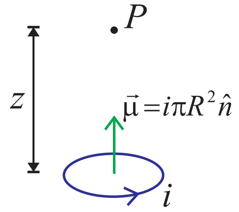
Recall from §7.1 Example 2, magnetic field at point P (height z above the ring)

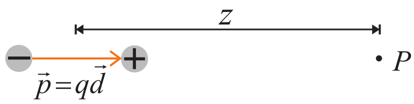


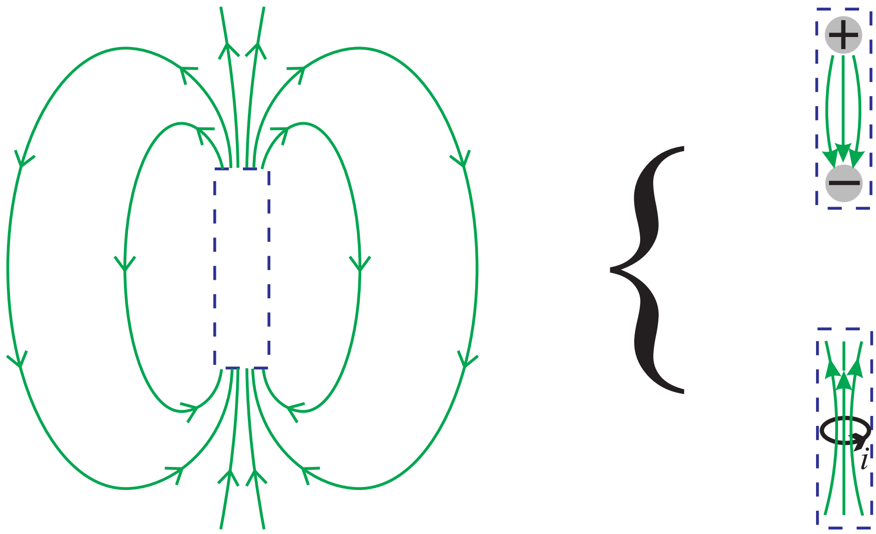
$$\vec{B} = \frac{\mu_0 i R^2 \hat{n}}{2(R^2 + z^2)^{3/2}} = \frac{\mu_0 \vec{\mu}}{2\pi(R^2 + z^2)^{3/2}}$$

At distance $z \gg R$,

$\vec{B} = \dfrac{\mu_0 \vec{\mu}}{2\pi z^3}$	$\vec{E} = \dfrac{\vec{p}}{4\pi\epsilon_0 z^3}$
due to *magnetic dipole*	due to *electric dipole*
(for $z \gg R$)	(for $z \gg d$)







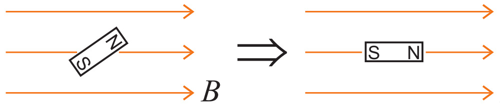
Also, notice $\vec{\mu}$ = magnetic dipole moment $\left[\begin{array}{ll}\text{Unit:} & \text{Am}^2 \\ & J/T\end{array}\right]$

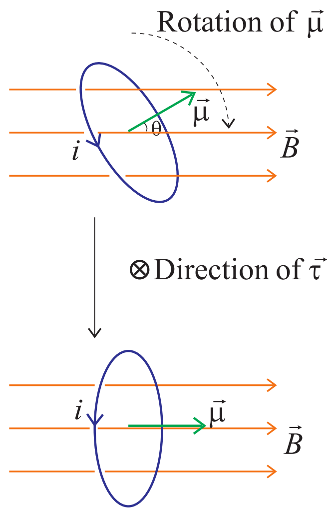
$$\begin{aligned} \mu_0 &= \text{Permeability of free space} \\ &= 4\pi \times 10^{-7}\text{Tm/A} \end{aligned}$$

7.5 Magnetic Dipole in A Constant B-field

In the presence of a constant magnetic field, we have shown for a *rectangular current loop*, it experiences a **torque** $\boxed{\vec{\tau} = \vec{\mu} \times \vec{B}}$. It applies to any magnetic dipole in general.

$\therefore$ External magnetic field aligns the magnetic dipoles.





Similar to electric dipole in a E-field, we can consider the work done in rotating the magnetic dipole. (refer to Chapter 2)

$$dW = -dU, \qquad \text{where } U \text{ is potential energy of dipole}$$

$$\boxed{U = -\vec{\mu} \cdot \vec{B}}$$

Note :

(1) We <u>cannot</u> define the potential energy of a magnetic field in general. However, we <u>can</u> define the potential energy of a magnetic dipole in a constant magnetic field.

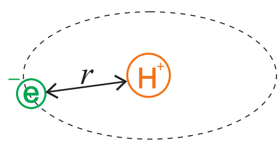
(2) In a *non-uniform external B-field*, the magnetic dipole will *experience a net force (not only net torque)*

7.6 Magnetic Properties of Materials

Recall intrinsic electric dipole in molecules:

Intrinsic dipole (magnetic) in atoms:

In our classical model of atoms, electrons revolve around a positive nuclear.



$$\therefore \quad \text{"Current" } i = \frac{e}{P}, \quad \text{where } P \text{ is period of one orbit around nucleus}$$

$$P = \frac{2\pi r}{v}, \quad \text{where } v \text{ is velocity of electron}$$

$\therefore$ **Orbit magnetic dipole** of atom:

$$\mu = iA = \left(\frac{ev}{2\pi r}\right)(\pi r^2) = \frac{erv}{2}$$

Recall: angular momentum of rotation $l = mrv$

$$\therefore \quad \mu = \frac{e}{2m} \cdot l$$

In *quantum mechanics*, we know that

$$l \text{ is quantized, i.e. } \quad l = N \cdot \frac{h}{2\pi}$$

$$\begin{aligned} \text{where } N &= \text{Any positive integer } (1,2,3, \ldots) \\ h &= \text{Planck's constant } (6.63 \times 10^{-34} J \cdot s) \end{aligned}$$

$\therefore$ Orbital magnetic dipole moment

$$\mu_l = \underbrace{\frac{eh}{4m\pi}}_{\textbf{Bohr's magneton } \mu_B = 9.27\times 10^{-24} J/T} \cdot N$$

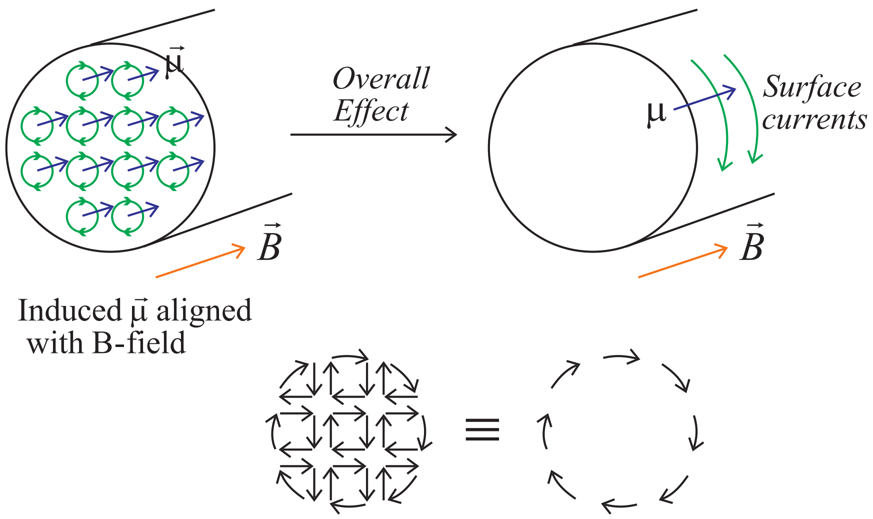
There is another source of intrinsic magnetic dipole moment inside an atom:

Spin dipole moment: coming from the intrinsic "spin" of electrons.

Quantum mechanics suggests that e^- are *always* spinning and it's either an "*up*" spin or a "*down*" spin

$$\mu_e = 9.65 \times 10^{-27} \approx \mu_B$$

So can there be induced magnetic dipole?



Recall: for electric field

$$\boxed{E_{dielectric} = K_e E_{vacuum}\ ; \quad K_e \geq 1}$$

For magnetic field in a material:

$$\vec{B}_{net} = \underset{\substack{\text{applied}\\ \text{B-field}}}{\underset{\uparrow}{\vec{B}_0}} + \underset{\substack{\text{B-field produced}\\ \text{by induced dipoles}}}{\underset{\uparrow}{\vec{B}_M}}$$

In many materials (except ferromagnets),

$$\vec{B}_M \propto \vec{B}_0$$

Define :

$$\boxed{\vec{B}_M = \chi_m \vec{B}_0}$$

χ_m is a *number* called **magnetic susceptibility**.

$$\begin{aligned} \therefore \quad \vec{B}_{net} &= \vec{B}_0 + \chi_m \vec{B}_0 \\ &= (1 + \chi_m)\,\vec{B}_0 \end{aligned}$$

$$\boxed{\vec{B}_{net} = \kappa_m \vec{B}_0\ ; \quad \kappa_m = 1 + \chi_m}$$

Define : κ_m is a *number* called **relative permeability**.

One more term ......

Define : the **Magnetization** of a material:

$$\vec{M} = \frac{d\vec{\mu}}{dV} \qquad \text{where } \vec{\mu} \text{ is magnetic dipole moment, } V \text{ is volume}$$

(or, the net magnetic dipole moment per unit volume)

In most materials (except ferromagnets),

$$\boxed{\vec{B}_M = \mu_0 \vec{M}}$$

Three types of magnetic materials:

(1) **Paramagnetic**:

$$\begin{matrix} \kappa_m \geq 1 \\ (\chi_m \geq 0) \end{matrix}\ , \quad \text{induced magnetic dipoles } \textit{aligned} \text{ } \textit{with} \text{ the applied B-field.}$$

`e.g.` Al ($\chi_m \doteq 2.2 \times 10^{-5}$), Mg ($1.2 \times 10^{-5}$), O_2 (2.0×10^{-6})

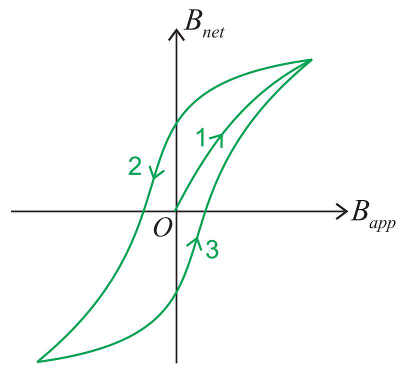
(2) **Diamagnetic**:

$$\begin{array}{c} \kappa_m \leq 1 \\ (\chi_m \leq 0) \end{array}, \quad \begin{array}{l} \text{induced magnetic dipoles } \textit{aligned} \\ \textit{opposite with} \text{ the applied B-field.} \end{array}$$

`e.g.` Cu ($\chi_m \approx -1 \times 10^{-5}$), Ag ($-2.6 \times 10^{-5}$), N_2 (-5×10^{-9})

(3) **Ferromagnetic**:

`e.g.` Fe, Co, Ni
Magnetization not linearly proportional to applied field.
$\Rightarrow \dfrac{B_{net}}{B_{app}}$ not a constant (can be as big as $\sim 5000 - 100,000$)



hysteresis curve
(hysteros: [Greek!] later, behind)

Interesting Case : *Superconductors*

$$\chi_m = -1$$

A perfect diamagnetic.
NO magnetic field inside.