MATRICES :—

1. INTRODUCTION :—

In this section, we are going to discuss a combination of algebra and geometry which is important in many applications. Problems in various fields of science and mathematics involve the solution of sets of linear equations, say for example – solve $x+2y=2$ & $2x+3y=6$. We can solve this easily and can find that, $x=6$ & $y=-2$ as the solution to these equations. This sounds like algebra, but it has an useful geometric interpretation. We can think of $x=6$, $y=-2$ as the point $(6,-2)$ in the (x,y) plane. Since, two linear equations represent two straight lines, the solution is then the point of intersection of the lines. The geometry helps us to understand that sometimes there is no solution (parallel lines) and sometimes there are infinitely many solutions (both equations represent the same line).

The language of "vectors" is very useful in studying sets of simultaneous equations. We are familiar with quantities such as the velocity ($\vec{v}$) of an object, the force ($\vec{F}$) acting on it, or the magnetic field ($\vec{B}$) at a point, which have both magnitude and direction. Such quantities are called "vectors"; contrast to them some quantities like mass, time or temperature, which have magnitude only, are called "scalars".

For example, for a particular Cartesian coordinate system, the components of the vector $\vec{a}$ will be $a_x\hat{i}, a_y\hat{j}, a_z\hat{k}$, and the vector $\vec{a}$ can be written as:

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}.$$

Although, here we've considered only real three-dimensional space, we may extend our notion of a vector to more abstract spaces, which in general can have an arbitrary number of dimensions N.

2. Vector spaces :—

A set of objects (vectors) $\vec{a}, \vec{b}, \vec{c}, \ldots$ is said to form a "linear vector space" V if:

(i) the set is closed under commutative and associative addition –

$$\vec{a}+\vec{b} = \vec{b}+\vec{a}$$

$$(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c}).$$

(ii) the set is closed under multiplication by a scalar (in general any complex number) to form a new vector $\lambda\vec{a}$, the operation being both distributive and associative, so that –

$$\left.\begin{aligned} \lambda(\vec{a}+\vec{b}) &= \lambda\vec{a}+\lambda\vec{b} \\ (\lambda+\mu)\vec{a} &= \lambda\vec{a}+\mu\vec{a} \end{aligned}\right\} \text{distributive}$$

$$\lambda(\mu\vec{a}) = (\lambda\mu)\vec{a} \longrightarrow \text{associative.}$$

where λ and μ are arbitrary scalars.

(iii) there exists a "null vector" $\vec{0}$, such that, $\vec{a}+\vec{0}=\vec{a}$ for all $\vec{a}$.

(iv) multiplication by "unity" leaves any vector unchanged: $1\times\vec{a}=\vec{a}$.

(v) all vectors have a corresponding "negative vector" $-\vec{a}$ such that, $\vec{a}+(-\vec{a})=\vec{0}$.

- If we restrict all scalars to be real, then we obtain a "real vector space" (example: our familiar three-dimensional space). Otherwise, in general, we obtain a "complex vector space".

- The "span" of a set of vectors $\vec{a},\vec{b},\ldots,\vec{s}$ is defined as the set of all vectors that may be written as a linear sum of the original set, i.e. all vectors

$$\vec{x} = \alpha\vec{a}+\beta\vec{b}+\cdots+\sigma\vec{s}$$

that result from the infinite number of possible values of the scalars $\alpha,\beta,\ldots,\sigma$.

That is, a collection/set of vectors is said to span the space if every vector can be written as a linear combination of the members of this set. A set of vectors that spans the space is also called "complete".

- If, for some choice of $\alpha,\beta,\ldots,\sigma$ (other than the trivial case, all zero), we get

$$\boxed{\alpha\vec{a}+\beta\vec{b}+\cdots+\sigma\vec{s}=\vec{0}}$$

, then the set of vectors $\vec{a},\vec{b},\ldots,\vec{s}$ is said to be "linearly dependent". If for any choice of $\alpha,\beta,\ldots,\sigma$ (other than all zero), we get $\boxed{\alpha\vec{a}+\beta\vec{b}+\cdots+\sigma\vec{s}\neq 0}$, then the vectors are called "linearly independent."

- Basis Vectors:

A set of linearly independent vectors that spans the space is called a "basis". The number of vectors in any basis is called the "dimension" of the space. For this chapter, we'll assume that the dimension is finite.

Suppose, a set of N linearly independent vectors $\hat{e}_1,\hat{e}_2,\ldots,\hat{e}_N$ forms a basis for N-dim. vector space. Then we can write any arbitrary vector $\vec{x}$ in this vector space as:

$$\boxed{\vec{x} = x_1\hat{e}_1+x_2\hat{e}_2+\cdots+x_N\hat{e}_N = \sum_{i=1}^{N} x_i\hat{e}_i}$$

for some set of coefficients x_i.

The coefficients x_i are the "components" of $\vec{x}$ with respect to the $\hat{e}_i$-basis.

Ex: Profe that, these components are "unique".

Proof: Let's assume, there exist another set of components such that,

$$\vec{x} = \sum_{i=1}^{N} x_i \hat{e}_i \text{ and also, } \vec{x} = \sum_{i=1}^{N} y_i \hat{e}_i .$$

then,
$$\sum_{i=1}^{N} (x_i - y_i)\hat{e}_i = 0$$

But, as $\hat{e}_i$ are linearly independent (by definition of basis vectors) this equation has only the solution $x_i = y_i$ for all $i = 1, 2, \dots, N$.

$\Rightarrow$ In fact, any set of N linearly independent vectors can form a basis for an N-dimensional space, if it spans the space.

We can always choose a different set of basis vectors $\hat{e}_i'$, $i = 1, 2, \dots, N$, so that we can write any vector $\vec{x}$ (of that vector space) as:

$$\vec{x} = x_1'\hat{e}_1' + x_2'\hat{e}_2' + \dots + x_N'\hat{e}_N' = \sum_{i=1}^{N} x_i'\hat{e}_i'$$

$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$ in cartesian

[e.g. $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\,\dot{\phi}\hat{\phi}$] in spherical-polar

(as, $d\vec{r} = dr\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$ (as $\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$)

- **Inner product:**

The "inner product" of two vectors is a generalisation of the dot-product (in real three-dim. space) to more abstract vector spaces. It is denoted in general by $\langle \vec{a} | \vec{b} \rangle$, which is a scalar function of $\vec{a}$ and $\vec{b}$.

The inner product has the following properties:

(i) $\langle \vec{a} | \vec{b} \rangle = \langle \vec{b} | \vec{a} \rangle^*$

(ii) $\langle \vec{a} | \lambda\vec{b} + \mu\vec{c} \rangle = \lambda \langle \vec{a} | \vec{b} \rangle + \mu \langle \vec{a} | \vec{c} \rangle$

From (i) & (ii) we get:

$$\langle \lambda\vec{a} + \mu\vec{b} | \vec{c} \rangle = \lambda^* \langle \vec{a} | \vec{c} \rangle + \mu^* \langle \vec{b} | \vec{c} \rangle$$

and,
$$\langle \lambda\vec{a} | \mu\vec{b} \rangle = \lambda^*\mu \langle \vec{a} | \vec{b} \rangle$$

- Two vectors in a general vector space are defined to be "orthogonal" if $\langle \vec{a} | \vec{b} \rangle = 0$.
- The "norm" of a vector $\vec{a}$ is given by $\|\vec{a}\| = \sqrt{\langle \vec{a} | \vec{a} \rangle}$.

  We'll only consider $\langle \vec{a} | \vec{a} \rangle \geq 0$

  $\langle \vec{a} | \vec{a} \rangle = 0$ if and only if $\vec{a} = \vec{0}$.

- One important property of basis vectors is that, they are "orthonormal":

$$\langle \hat{e}_i | \hat{e}_j \rangle = \delta_{ij}$$

$\hat{e}_i$ are basis vectors of that vector space.

[$\delta_{ij} = 1$ for $i = j$
$= 0$ for $i \neq j$]

"Kronecker delta"

- In the above basis we may express any two vectors $\vec{a}$ and $\vec{b}$ as –

$$\boxed{\vec{a} = \sum_{i=1}^{N} a_i \hat{e}_i \quad \text{and} \quad \vec{b} = \sum_{i=1}^{N} b_i \hat{e}_i}$$

for any $\vec{a}$, we have

$$\langle \hat{e}_j | \vec{a} \rangle = \sum_{i=1}^{N} \langle \hat{e}_j | a_i \hat{e}_i \rangle = \sum_{i=1}^{N} a_i \langle \hat{e}_j | \hat{e}_i \rangle = a_j$$

(as $\langle \hat{e}_j | \hat{e}_i \rangle = \delta_{ij}$)

Thus, the components of any vector $\vec{a}$ is given by $a_i = \langle \hat{e}_i | \vec{a} \rangle$.

The inner product of $\vec{a}$ and $\vec{b}$ can be written in terms of their components in an orthonormal basis as –

$$\langle \vec{a} | \vec{b} \rangle = \langle \sum_{i=1}^{N} a_i \hat{e}_i \Big| \sum_{j=1}^{N} b_j \hat{e}_j \rangle$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} a_i^* b_j \underbrace{\langle \hat{e}_i | \hat{e}_j \rangle}_{= \delta_{ij}}$$

$$= \sum_{i=1}^{N} a_i^* b_i$$

These clearly are the generalizations of the dot products of vectors in real three-dimensional space; where the unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$ form basis. Any vectors $\vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$ and $\vec{b} = b_x \hat{i} + b_y \hat{j} + b_z \hat{k}$, then, $\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$

$$\vec{a} \cdot \vec{a} = a_x^2 + a_y^2 + a_z^2$$

and $a_x = \hat{i} \cdot \vec{a}$, $a_y = \hat{j} \cdot \vec{a}$, $a_z = \hat{k} \cdot \vec{a}$.

- Linear Operators:

A linear operator "operates" on some vector $\vec{x}$ and give another vector $\vec{y}$ i.e. $\vec{y} = \mathcal{A} \vec{x}$ in such a way that, for any two vectors $\vec{a}$ and $\vec{b}$,

$$\mathcal{A}(\lambda \vec{a} + \mu \vec{b}) = \lambda \mathcal{A} \vec{a} + \mu \mathcal{A} \vec{b} \longrightarrow \text{"linear transformation"}$$

where λ, μ are scalars.

$\vec{x}$ & $\vec{y}$ may not be in the same vector space

- If we know what a particular linear transformation does to a set of basis vectors, we can then easily figure out what it does to any vector.

Suppose, $\mathcal{A} \hat{e}_j = \sum_{i=1}^{N} A_{ij} \hat{e}_i$.

A_{ij} is the ith component of the vector $\mathcal{A}\hat{e}_j$ in this basis; collectively the numbers A_{ij} are called "components" of the linear operator $\mathcal{A}$ in the $\hat{e}_i$-basis.

Then we can express the relation $\vec{y} = A\vec{x}$ in component form as:

$$\vec{y} = \sum_{i=1}^{N} y_i \hat{e}_i = A\left(\sum_{j=1}^{N} x_j \hat{e}_j\right)$$

$$\Rightarrow \sum_{i=1}^{N} y_i \hat{e}_i = \sum_{j=1}^{N} x_j \sum_{i=1}^{N} A_{ij} \hat{e}_i$$

Hence, in purely component form, we have:

$$\boxed{y_i = \sum_{j=1}^{N} A_{ij} x_j \equiv \vec{y} = A\vec{x}}$$

• Properties of linear operators:

If $\vec{x}$ is a vector and A and B are two linear operators, then,

$$(A+B)\vec{x} = A\vec{x} + B\vec{x} \longrightarrow \text{distributive}$$

$$(\lambda A)\vec{x} = \lambda(A\vec{x})$$

$$(AB)\vec{x} = A(B\vec{x}) \longrightarrow \text{"associative"}$$

In general, $AB\vec{x} \neq BA\vec{x} \rightarrow$ Non-commutative

The null (zero) and identity operators are -

$$\boxed{O\vec{x} = \vec{0} \quad \text{and} \quad I\vec{x} = \vec{x}}$$

Two operators A and B are equal if $A\vec{x} = B\vec{x}$ for all vectors $\vec{x}$.

There exists an operator A^{-1} such that,

$\boxed{AA^{-1} = A^{-1}A = I}$, then A^{-1} is the inverse of A.

⇒ Some linear operators do not possess an inverse and they are called "singular", while those operators that do have an inverse are called "non-singular".

③ Matrices :—

We have seen that, in a particular basis $\hat{e}_i$, both vectors and linear operators can be expressed in terms of their components with respect to the basis. These components may be displayed as an array of numbers, ~~they are~~ it is called "matrix".

In general, if a linear operator A transforms vectors ($\vec{x}$) from an N-dimensional vector space into another vectors ($\vec{y}$) from an M-dimensional space, then we may represent the operator ~~A~~ by the matrix:

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & & \vdots \\ A_{M1} & A_{M2} & \cdots & A_{MN} \end{pmatrix}$$

$$A\vec{x} = \vec{y} \Rightarrow y_i = \sum_{j=1}^{N} A_{ij} x_j$$

↳ $M \times N$ matrix — M rows and N columns.

If the dimensions of the two vector spaces are the same, the matrix is represented as $N \times N$ matrix or a "square matrix" of order N.

In a similar way, we may denote a vector $\vec{x}$ in terms of its components x_i in a basis $\hat{e}_i$, $i = 1, 2, \cdots N$, by the array-

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \qquad \vec{x} = \sum_{i=1}^{N} x_i \hat{e}_i .$$

↳ which is called a "column matrix" (or a column vector)

⇒ Note that, in a different basis $\hat{e}'_i$, the same vector $\vec{x}$ would be represented by a different column matrix containing the components x'_i in the new basis, i.e.

$$x' = \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_N \end{pmatrix} \qquad \vec{x} = \sum_{i=1}^{N} x'_i \hat{e}'_i .$$

3.1. Basic Matrix Algebra:

The basic algebra of matrices can be deduced from the properties of the linear operators that they represent.

$$(A+B)_{ij} = A_{ij} + B_{ij}$$

$$(\lambda A)_{ij} = \lambda A_{ij}$$

$$(AB)_{ij} = \sum_k A_{ik} B_{kj} \Longleftrightarrow \left[\sum_j^{\text{As}'} (AB)_{ij} x_j = \sum_k A_{ik} (Bx)_k = \sum_j \sum_k A_{ik} B_{kj} x_j \right]$$

- **Matrix addition and multiplication:**

(i) The sum of two matrices with same dimension $(M \times N)$ is a matrix of same dimension $(M \times N) \Rightarrow S = A + B$

$$S_{ij} = A_{ij} + B_{ij} \quad \Big\| \text{ OR, difference, } D = A - B, \; D_{ij} = A_{ij} - B_{ij} \Big\|$$

If A & B are 2×3 matrices;

$$\begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} + B_{11} & A_{12} + B_{12} & A_{13} + B_{13} \\ A_{21} + B_{21} & A_{22} + B_{22} & A_{23} + B_{23} \end{pmatrix}.$$

- $A + B = B + A \longrightarrow$ matrix addition is "commutative".
- $A + (B + C) = (A + B) + C \longrightarrow$ "associative".

(ii) matrix multiplication with a scalar —

$$(\lambda A)_{ij} = \lambda A_{ij}$$

$$\lambda \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{pmatrix} = \begin{pmatrix} \lambda A_{11} & \lambda A_{12} & \lambda A_{13} \\ \lambda A_{21} & \lambda A_{22} & \lambda A_{23} \end{pmatrix}$$

Ex: $A = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$, $C = \begin{pmatrix} -2 & 1 \\ -1 & 1 \end{pmatrix}$

Find the matrix $D = A + 2B - C$.

ans: $$D = \begin{pmatrix} 2 + 2\times 1 - (-2) & -1 + 2\times 0 - 1 \\ 3 + 2\times 0 - (-1) & 1 + 2\times(-2) - 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & -2 \\ 4 & -4 \end{pmatrix}.$$

(iii) Matrix multiplication with another matrix:

Let us consider, $\vec{y} = A\vec{x}$

$$\Rightarrow y_i = \sum_{j=1}^{N} A_{ij} x_j, \text{ for } i = 1, 2, \dots, M.$$

in matrix form, we have $y = Ax$

$$\Rightarrow \begin{pmatrix} \boxed{y_1} \\ y_2 \\ \vdots \\ y_M \end{pmatrix} = \begin{pmatrix} \boxed{A_{11} \quad A_{12}, \cdots \quad A_{1N}} \\ A_{21} \quad A_{22} \quad \cdots \quad A_{2N} \\ \vdots \\ A_{M1} \quad A_{M2} \quad \cdots \quad A_{MN} \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{matrix}} \end{pmatrix}$$

If we want to calculate y_1,

$$y_1 = A_{11}x_1 + A_{12}x_2 + \cdots + A_{1N}x_N$$

all other components y_i are calculated similarly.

Similarly, we can calculate $P = AB$, where P is the matrix of quantities formed by the operation of the rows of A on the columns of B, treating each column of B in turn as the vector $\vec{x}$ represented in component form.

$\Rightarrow$ The number of columns in A must equal to the number of rows in B.

$$\underset{M \times R}{P} = \underset{M \times \underline{\underline{N}}}{A} \cdot \underset{\underline{\underline{N}} \times R}{B}$$

$$P_{ij} = \sum_{k=1}^{N} A_{ik} B_{kj}, \text{ for } i = 1, 2, \dots, M, \; j = 1, 2, \dots, R.$$

For example: $P = AB$ for A being 2×3 and B being 3×2 matrix.

$$\begin{pmatrix} \boxed{P_{11}} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \begin{pmatrix} \boxed{A_{11} \quad A_{12} \quad A_{13}} \\ A_{21} \quad A_{22} \quad A_{23} \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} B_{11} \\ B_{21} \\ B_{31} \end{matrix}} & \begin{matrix} B_{12} \\ B_{22} \\ B_{32} \end{matrix} \end{pmatrix}$$

where,
$$P_{11} = A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31}$$
$$P_{21} = A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31}$$
$$P_{12} = A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32}$$
$$P_{22} = A_{21}B_{12} + A_{22}B_{22} + A_{23}B_{32}$$

- $A(BC) \equiv (AB)C \longrightarrow$ matrix multiplication is "associative". (provided all the products are defined)

- In general, $AB \neq BA \longrightarrow$ ~~matrix~~ multiplication of matrices is not, in general, commutative.

As, say $P = AB$ & $Q = BA$, if A is an $M \times N$ matrix and B is an $N \times M$ matrix, then P is an $M \times M$ matrix, whilst Q is an $N \times N$ matrix. Even if both A and B are square matrices, $AB \neq BA$.

Ex: Evaluate $P = AB$ and $Q = BA$, where

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 0 & 3 & 2 \\ 1 & -3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$$

Ans: $P = \begin{pmatrix} 5 & -6 & 8 \\ 9 & 7 & 2 \\ 11 & 3 & 7 \end{pmatrix}, \quad Q = \begin{pmatrix} 9 & -11 & 6 \\ 3 & 5 & 1 \\ 10 & 9 & 5 \end{pmatrix}, \quad P \neq Q.$

- Matrix multiplication is "distributive" over addition

$$(A+B)C = AC + BC.$$

and

$$C(A+B) = CA + CB.$$

We just have to maintain the order of multiplications, as $AB \neq BA$.

- Define the "commutator" of A and B as – $\boxed{[A, B] = AB - BA}$

- The null and identity matrices:

The "null" or "zero" matrix O has all elements equal to zero, and its properties are -

$$AO = O = OA$$
$$A + O = O + A = A.$$

The "identity" matrix I has the property

$$AI = IA = A.$$

$\Rightarrow$ As both null & identity matrices have the property of commutative multiplication, they both must be "square" matrices.

The $N \times N$ identity matrix (I_N) has the form

$$I_N = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

- Functions of matrices:

$$A^n = AA \ldots A \text{ (n times)}$$

in general, we can construct $\Rightarrow S = \sum_n a_n A^n$

a_n ↳ simple scalars.

$$\exp A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

$$\sin A = \sum_{n=0}^{\infty} (-1)^n \frac{A^{2n+1}}{(2n+1)!}$$

$$\cos A = \sum_{n=0}^{\infty} (-1)^n \frac{A^{2n}}{(2n)!}$$

Transpose of matrix :

One can form a matrix by interchanging the rows and columns.

Transpose of $A = A^T$

Ex: Find the transpose of $A = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 4 & 1 \end{pmatrix}$

$$A^T = \begin{pmatrix} 3 & 0 \\ 1 & 4 \\ 2 & 1 \end{pmatrix}$$

If A is an $M \times N$ matrix, then A^T is an $N \times M$ matrix.

Transpose of a column matrix is a row matrix.

- $(AB)^T = B^T A^T$

proof: $(AB)^T_{ij} = (AB)_{ji} = \sum_k A_{jk} B_{ki}$

$$= \sum_k (A^T)_{kj} (B^T)_{ik} = \sum_k (B^T)_{ik} (A^T)_{kj}$$

$$= (B^T A^T)_{ij}$$

in general, $(ABC \cdots G)^T = G^T \cdots C^T B^T A^T$

The complex and Hermitian conjugates of a matrix :

(i) "Complex ~~eq~~ conjugate" of matrix A is the matrix obtained by taking the complex conjugate of the each elements of A

$$(A^*)_{ij} = (A_{ij})^*$$

If the matrix is real, then $A^* = A$.

Ex: $A = \begin{pmatrix} 1 & 2 & 3i \\ 1+i & 1 & 0 \end{pmatrix}$, so, $A^* = \begin{pmatrix} 1 & 2 & -3i \\ 1-i & 1 & 0 \end{pmatrix}$.

(ii) "Hermitian conjugate" or "adjoint" of a matrix A is the transpose of its complex conjugate, or equivalently the complex conjugate of its transpose,

$$A^\dagger = (A^*)^T = (A^T)^*$$

If the matrix is real, then $A^\dagger = A^T$

In can be shown that, $(AB \cdots G)^\dagger = G^\dagger \cdots B^\dagger A^\dagger$

Ex: $A = \begin{pmatrix} 1 & 2 & 3i \\ 1+i & 1 & 0 \end{pmatrix}$, then $A^\dagger = \begin{pmatrix} 1 & 1-i \\ 2 & 1 \\ -3i & 0 \end{pmatrix}$

$\Rightarrow$ The Hermitian conjugate is an important quantity as it is connected with the inner product of two vectors.

Suppose, two vectors $\vec{a}$ and $\vec{b}$ are represented by column matrices,

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix}$$

$$a^\dagger b = (a_1^* \; a_2^* \cdots a_N^*) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix} = \sum_{i=1}^{N} a_i^* b_i = \langle \vec{a} | \vec{b} \rangle$$

in that basis.

- **The trace of a matrix:**

defined as the sum of the diagonal elements of a square matrix.

$$\mathrm{Tr}\,A = A_{11} + A_{22} + \cdots + A_{NN} = \sum_{i=1}^{N} A_{ii}$$

Defined only for square matrices.

$\Rightarrow$ Taking the trace is a linear operation —

$$\mathrm{Tr}(A \pm B) = \mathrm{Tr}A \pm \mathrm{Tr}B$$

$\Rightarrow$ Trace of the product of two matrices is independent of the order of their multiplication, even if the matrices do not commute.

Proof: $\mathrm{Tr}\,AB = \sum_{i=1}^{N} (AB)_{ii} = \sum_{i=1}^{N}\sum_{j=1}^{N} A_{ij} B_{ji} = \sum_{i=1}^{N}\sum_{j=1}^{N} B_{ji} A_{ij} = \sum_{j=1}^{N} (BA)_{jj}$

$= \mathrm{Tr}\,BA.$

$\Rightarrow$ $\mathrm{Tr}\,ABC = \mathrm{Tr}\,BCA = \mathrm{Tr}\,CAB \rightarrow$ invariant under "cyclic permutations" of the matrices in the product.

$\Rightarrow$ $\mathrm{Tr}\,A^T = \mathrm{Tr}\,A$

$\mathrm{Tr}\,A^\dagger = (\mathrm{Tr}\,A)^*$

- **The determinant of a matrix:** a single number that depends upon the elements of that matrix.

If A is a 3×3 matrix, then its determinant, of order 3, is denoted by

$$\det A = |A| = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$$

defined only for square matrices.

In order to calculate the value of a determinant, let's first introduce two terms: "minor" & "cofactor".

- The minor M_{ij} of the element A_{ij} of an $N\times N$ matrix is the determinant of the $(N-1)\times(N-1)$ matrix obtained by removing entire i^{th} row and j^{th} column of A.
- The cofactor C_{ij} is basically $(-1)^{i+j} M_{ij}$

Ex: Find the cofactor of the element A_{23} of the matrix

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

minor, $M_{23} = \begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix}$

so, $C_{23} = (-1)^{2+3} \begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix} = - \begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix}$

• We now define a determinant as the sum of the products of the elements of any row or column and their corresponding cofactors.

Such sum is called a "Laplace Expansion".

Ex

$|A| = A_{11}C_{11} + A_{12}C_{12} + A_{13}C_{13} \longrightarrow$ expanding in terms of 1st row

$= A_{21}C_{21} + A_{22}C_{22} + A_{23}C_{23} \longrightarrow$ " " 2nd row

$= A_{13}C_{13} + A_{23}C_{23} + A_{33}C_{33} \longrightarrow$ " " " 3rd column.

⋮ etc.

↳ they all are the same – the value of the determinant is independent of the row or column chosen.!

$$|A| = A_{21}(-1)^{(2+1)}M_{21} + A_{22}(-1)^{2+2}M_{22} + A_{23}(-1)^{2+3}M_{23}$$

$$= -A_{21}\begin{vmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{vmatrix} + A_{22}\begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} - A_{23}\begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix}$$

We can then also evaluate the values of individual determinants by using the same rule.

like, $\begin{vmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{vmatrix} = A_{12}(-1)^{1+1}M_{11} + A_{13}(-1)^{1+2}M_{12}$

$\cancel{= A_{12}A_{33} + A_{13}}$

$= A_{12}A_{33} - A_{13}A_{32}$

so, $|A| = -A_{21}(A_{12}A_{33} - A_{13}A_{32}) + A_{22}(A_{11}A_{33} - A_{13}A_{31})$
$- A_{23}(A_{11}A_{32} - A_{12}A_{31}) \longrightarrow$ Expanding using 2nd row

$= A_{11}(A_{22}A_{33} - A_{23}A_{32}) + A_{12}(A_{23}A_{31} - A_{21}A_{33})$
$+ A_{13}(A_{21}A_{32} - A_{22}A_{31}) \longrightarrow$ Expanding using 1st row

Ex: Suppose the rows of a real 3×3 matrix A are interpreted as the components, in a given basis, of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

i.e. $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$

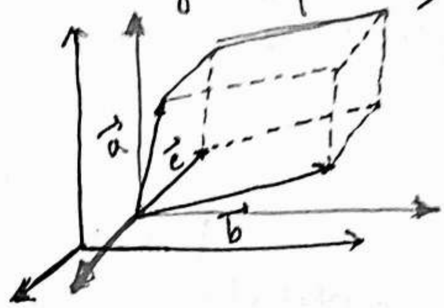
Then the det (A),

$$|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$

$$\Rightarrow |A| = \vec{a} \cdot (\vec{b} \times \vec{c})$$

In other words, $|A|$ is the volume of the parallelepiped defined by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ $\Rightarrow$ Geometrical interpretation.

So, if the vectors $\vec{a}, \vec{b}, \vec{c}$ are not linearly independent, then the determinant vanishes, $|A| = 0$



- Properties of determinants:

[The proofs can be follwed from the alternative form for a determinant expressed in terms of the Levi-Civita symbol $\epsilon_{ijk\cdots}$.

For example: $|A| = \begin{vmatrix} a_1 & a_2 & \cdots & a_N \\ b_1 & b_2 & \cdots & b_N \\ c_1 & c_2 & \cdots & c_N \\ \vdots & \vdots & & \vdots \end{vmatrix} = \sum_{i,j,k,\ldots}^{N} \epsilon_{ijk\ldots} a_i b_j c_k \cdots$

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } i,j,k \text{ is an even permutation of } 1,2,3 \\ -1 & \text{if } i,j,k \text{ is an odd permutation of } 1,2,3 \\ 0 & \text{otherwise / any indices repeated.} \end{cases}$$

But, here we will present all the ~~proofs~~ properties without proofs.]

(i) Determinant of transpose:

$$|A^T| = |A|$$

(ii) Determinant of complex and Hermitian conjugate:

$$|A^\dagger| = |(A^*)^T| = |A^*| = |A|^*$$

(iii) If two rows or two columns are interchanged, its determinant changes sign, but the ~~vale~~ magnitude remains the same.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = - \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

(iv) If all elements of a single row or single column have a common factor say λ, then this factor can be taken outside, i.e.

$$\begin{vmatrix} \lambda a_1 & \lambda a_2 & \lambda a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \lambda \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

So, If all elements of any row or any column are zero, then $|A|=0$.

If every element of an $N \times N$ matrix A is multiplied by a constant factor λ, then $|\lambda A| = \lambda^N |A|$.

(v) Identical rows or columns: If any two rows or any two columns are identical or are multiples of one another, then $|A|=0$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0 \quad \text{or,} \quad \begin{vmatrix} a_1 & a_2 & a_1 \\ b_1 & b_2 & b_1 \\ c_1 & c_2 & c_1 \end{vmatrix} = 0$$

$$\text{or,} \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \lambda a_1 & \lambda a_2 & \lambda a_3 \end{vmatrix} = 0.$$

(vi) Adding or substracting a constant multiple of one row/column to or from another: the determinant will remain unchanged.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 \pm \lambda a_2 & a_2 & a_3 \\ b_1 \pm \lambda b_2 & b_2 & b_3 \\ c_1 \pm \lambda c_2 & c_2 & c_3 \end{vmatrix}$$

(vii) Determinant of a product:

$$|AB| = |A|\,|B| = |BA|$$

in general, $|ABC| = |BAC| = |ACB| = |BCA| = \cdots$ invariant under any permutation of them.

$\Rightarrow$ A matrix whose determinant is zero is called a "singular matrix", otherwise it is "non-singular".

• Inverse of a matrix : | defined only for square matrices..

If A is a non-singular matrix, we can define a matrix, denoted by A^{-1}, called the inverse of A, which has the property that, if $AB = P$, then $B = A^{-1}P$ (note the ordering, it's not PA^{-1})

It is clear that, $A^{-1}A = I = AA^{-1}$, where I is the unit matrix.

To find A^{-1},

$$A^{-1} = \frac{C^T}{|A|} = \frac{\text{Transpose of the cofactor matrix}}{\det |A|}$$

for a 2×2 matrix,

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

so, $|A| = A_{11}A_{22} - A_{12}A_{21}$, And the matrix of cofactors is

$$C = \begin{pmatrix} A_{22} & -A_{21} \\ -A_{12} & A_{11} \end{pmatrix}, \quad C^T = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$$

$$A^{-1} = \frac{C^T}{|A|} = \frac{1}{A_{11}A_{22} - A_{12}A_{21}} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$$

Example: Find the inverse of the matrix

$$A = \begin{pmatrix} 2 & 4 & 3 \\ 1 & -2 & -2 \\ -3 & 3 & 2 \end{pmatrix}$$

$$|A| = 11, \quad C = \begin{pmatrix} 2 & 4 & -3 \\ 1 & 13 & -18 \\ -2 & 7 & -8 \end{pmatrix} \text{ and } C^T = \begin{pmatrix} 2 & 1 & -2 \\ 4 & 13 & 7 \\ -3 & -18 & -8 \end{pmatrix}$$

$$\text{so, } A^{-1} = \frac{C^T}{|A|} = \frac{1}{11} \begin{pmatrix} 2 & 1 & -2 \\ 4 & 13 & 7 \\ -3 & -18 & -8 \end{pmatrix}.$$

• properties of matrices :

(i) $(A^{-1})^{-1} = A$.

(ii) $(A^T)^{-1} = (A^{-1})^T$

(iii) $(A^\dagger)^{-1} = (A^{-1})^\dagger$

(iv) $(AB)^{-1} = B^{-1}A^{-1}$

(v) $(AB\ldots G)^{-1} = G^{-1}\ldots B^{-1}A^{-1}$.

4. Special types of square matrix :—

(a) Diagonal matrices:

having non-zero elements only on leading diagonal.
i.e. only elements A_{ij} with $i=j$ may be non-zero. (or some of them may be zero)

$$A_{ij} = 0 \text{ if } i \neq j \Rightarrow$$ these matrices often are denoted by diag(-,-,--)

- If a matrix has the form $A = \text{diag}(A_{11}, A_{22}, \dots, A_{NN})$, [N×N matrix]

$$|A| = A_{11} A_{22} \cdots A_{NN}$$

$$A^{-1} = \text{diag}\left(\frac{1}{A_{11}}, \frac{1}{A_{22}}, \dots, \frac{1}{A_{NN}}\right).$$

$$= \begin{pmatrix} \frac{1}{A_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{A_{22}} & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \frac{1}{A_{NN}} \end{pmatrix}$$

- If A and B are both diagonal, then $AB = BA$, which is not true for other matrices in general.

(b) Lower and Upper triangular matrices:

If all the elements of a square matrix above its principal diagonal are zero ⟶ lower triangular matrix.

Example in 3X3: $A = \begin{pmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$, where the elements $A_{ij} = 0$ for $j > i$

and A_{ij} may or may not $\neq 0$ for $j \leq i$

Similarly an upper triangular matrix has all zero elements below its principal diagonal.

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix}$$

- The determinant of an upper or lower triangular matrix is equal to the product of its diagonal elements (can be shown using Laplace expansion):

$$|A| = A_{11} A_{22} \cdots A_{NN}$$ for A being an N×N matrix.

- Inverse of a non-singular lower (upper) triangular matrix is also lower (upper) triangular matrix.

(c) Symmetric and anti-symmetric matrices:

- A square matrix A of order N is said to be "symmetric" if $\boxed{A = A^T}$.

- Similarly A is said to be "anti-symmetric" or "skew-symmetric" if $\boxed{A = -A^T}$ and its diagonal elements $(a_{11}, a_{22}, \cdots, a_{NN})$ are necessarily zero.

⇒ Moreover if A is symmetric or anti-symmetric, then its inverse A^{-1} will also be symmetric or anti-symmetric.

Proof: if $A = \pm A^T$, then

$$(A^{-1})^T = (A^T)^{-1} = \pm A^{-1}$$

⇒ Any $N \times N$ matrix A can be written as the sum of a symmetric and an antisymmetric matix.

Proof: We may write the matrix A as -

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

$$= B + C$$

where, clearly $B = B^T$ and $C = -C^T$. The matrix B is therefore called the symmetric part of A and C is the anti-symmetric part of A.

⇒ If A is an $N \times N$ anti-symmetric matrix, show that $|A| = 0$ if N is odd.

Proof: If A is anti-symmetric then $A^T = -A$.

taking determinant,

$$|A^T| = |-A|$$

$$\Rightarrow |A| = |-A| \quad [\text{as. } |A^T| = |A|]$$

$$= (-1)^N |A| \quad [\text{as } |\lambda A| = \lambda^N |A|]$$

Thus, if N is odd then $|A| = -|A|$, which implies that, $|A| = 0$

(d) Orthogonal matrices:

A non-singular matrix A will be called an "orthogonal matrix" if

$$\boxed{A^T = A^{-1}}$$

or $\boxed{AA^T = I}$ (A special case of unitary matrix)

⇒ Inverse of an orthogonal matrix is also orthogonal.

Proof: Say A is an orthogonal matrix, then

$$(A^{-1})^T = (A^T)^{-1} = (A^{-1})^{-1} \quad (\text{as } A^T = A^{-1} \text{ by def}^n)$$

So, A^{-1} is also orthogonal.

$\Rightarrow$ The determinant of an orthogonal matrix must be $|A| = \pm 1$

Proof: Since, $A^T A = I$

$$\Rightarrow |A^T A| = |I|$$
$$\Rightarrow |A^T||A| = 1 \quad [\text{as } |AB| = |A||B| = |B||A| = |BA|]$$
$$\Rightarrow |A|^2 = 1 \quad [\text{as } |A^T| = |A|]$$

So, $|A| = \pm 1$.

$\Rightarrow$ Action of a linear operator represented by an orthogonal matrix does not change the norm of a real vector.

Proof: Suppose that $\vec{y} = \mathcal{A}\vec{x}$ is represented, in some basis, by the matrix equation $y = Ax$ then $\langle \vec{y}|\vec{y}\rangle$ is given in that basis by

$$y^T y = (Ax)^T (Ax)$$
$$= x^T A^T A x \quad [\text{as } (AB)^T = B^T A^T]$$
$$= x^T x \quad [\text{as } A \text{ is an orthogonal matrix i.e. } A^T A = I]$$

So, $\langle \vec{y}|\vec{y}\rangle = \langle \vec{x}|\vec{x}\rangle \rightarrow$ norm (length) is unchanged.

(e) Hermitian and anti-Hermitian matrices:

- If $\boxed{A = A^\dagger} \rightarrow$ "Hermitian" matrix (where $A^\dagger$ is the Hermitian conjugate of A)
- If $\boxed{A = -A^\dagger} \rightarrow$ "anti-Hermitian" matrix.
- A real (anti-) symmetric matrix is a special case of an (anti-) Hermitian matrix, where all elements of that matrix are real.

$\Rightarrow$ If A is an (anti-) Hermitian matrix, then its inverse A^{-1} is also an (anti-) Hermitian matrix.

Proof: $(A^{-1})^\dagger = (A^\dagger)^{-1} = \pm A^{-1}$ [as $A = \pm A^\dagger$ by def^n]

$\Rightarrow$ Any $N \times N$ matrix A can be written as the sum of an Hermitian matrix and an anti-Hermitian matrix

Proof: $A = \frac{1}{2}(A + A^\dagger) + \frac{1}{2}(A - A^\dagger) = B + C$

where, $\underset{\text{Hermitian part}}{B = B^\dagger}$ and $C = -C^\dagger$ anti-Hermitian part

(f) Unitary matrices:

If $\boxed{A^{\dagger} = A^{-1}}$ $\longrightarrow$ Unitary matrix.

- Clearly, if A is real then $A^{\dagger} = A^{T}$, showing that a real orthogonal matrix is a special case of a Unitary matrix, where all elements of A are real.

$\Rightarrow$ Inverse of a unitary matrix is also unitary.

Proof: $$(A^{-1})^{\dagger} = (A^{\dagger})^{-1} = (A^{-1})^{-1}$$

$\Rightarrow$ Determinant of a unitary matrix has unit modulus

Proof: Since $A^{\dagger}A = I$

$$\Rightarrow |A^{\dagger}A| = |I| = 1.$$

$$\Rightarrow |A^{\dagger}|\,|A| = 1$$

$$\Rightarrow |A|^{*}|A| = 1 \quad [\text{as } |A^{\dagger}| = |A|^{*}]$$

$\Rightarrow$ The action of a linear operator represented by a unitary matrix does not change the norm of a complex vector.

Proof: If $\vec{y} = \mathcal{A}\vec{x}$ is represented in some basis by the matrix equation $y = Ax$, then $\langle \vec{y}|\vec{y}\rangle$ is given in that basis by [$\vec{x}$, $\vec{y}$ are "complex vectors"]

$$y^{\dagger}y = (Ax)^{\dagger}(Ax)$$
$$= x^{\dagger}A^{\dagger}Ax$$
$$= x^{\dagger}x \quad [\text{as } A^{\dagger}A = I]$$

Hence $\langle \vec{y}|\vec{y}\rangle = \langle \vec{x}|\vec{x}\rangle$

(g) Normal matrices:

If a matrix A commutes with its Hermitian conjugate, then A is called "normal matrix".

i.e. $\boxed{AA^{\dagger} = A^{\dagger}A.}$

$\Rightarrow$ The Hermitian and unitary matrices (or the symmetric and orthogonal matrices in real case) are example of normal matrices.

Proof: If A is Hermitian, $A = A^{\dagger}$

$$\text{So, } AA^{\dagger} = AA = A^{\dagger}A.$$

Similarly, for a unitary matrix, $A^{-1} = A^{\dagger}$

$$\text{So, } AA^{\dagger} = AA^{-1} = A^{-1}A = A^{\dagger}A.$$

⇛ If A is normal, then its inverse A^{-1} is also normal.

Proof: $A^{-1}(A^{-1})^{\dagger} = A^{-1}(A^{\dagger})^{-1} = (A^{\dagger}A)^{-1}$ [as $(AB)^{-1} = B^{-1}A^{-1}$]

$= (AA^{\dagger})^{-1}$ [as A is normal matrix]

$= (A^{\dagger})^{-1} A^{-1}$

$= (A^{-1})^{\dagger} A^{-1}$

So, **in short:**

real		Complex	
Symmetric:	$A = A^T$	Hermitian:	$A = A^{\dagger}$
anti-symmetric:	$A = -A^T$	anti-Hermitian:	$A = -A^{\dagger}$
Orthogonal:	$A^T = A^{-1}$	Unitary:	$A^{\dagger} = A^{-1}$

all are normal matrices

i.e. $AA^{\dagger} = A^{\dagger}A$

(5) Eigenvectors and eigenvalues:

Suppose that a linear operator $\mathcal{A}$ transforms a non-zero vector $\vec{x}$ in an N-dim. vector space into other vector $\mathcal{A}\vec{x}$ in the same space in this way –

$$\boxed{\mathcal{A}\vec{x} = \lambda\vec{x}} \qquad (\lambda \equiv \text{some scalar})$$

then $\vec{x}$ is called an "eigenvector" of the linear operator $\mathcal{A}$ and λ is called the corresponding "eigenvalue".

- In general, the operator $\mathcal{A}$ has N no. of independent eigenvectors $\vec{x}^i$, with eigenvalues λ_i (some of λ_i's might be same)

→ In matrix notation:

$$\boxed{Ax = \lambda x} \iff \boxed{(A - \lambda I)x = 0}$$

where A is an N×N matrix and x is a column matrix of N-dimension.

Clearly, if x is an eigenvector of A with some eigenvalue λ, then any scalar multiple μx will also be an eigenvector (as $A(\mu x) = \lambda(\mu x)$) with the same eigenvalue λ. We therefore often use "normalized" eigenvectors, for which

$$x^\dagger x = 1. \quad (\text{Like the norm or inner-product } \langle\vec{x}|\vec{x}\rangle \text{ in our basis})$$

Any eigenvector x can be normalized by dividing all its components by the scalar $(x^\dagger x)^{1/2}$

Ex: A non-singular matrix A has eigenvalues λ_i and eigenvectors x^i. Find the eigenvalues and eigenvectors of the inverse matrix A^{-1}.

Ans: $Ax^i = \lambda_i x^i$

Lef-multiplying both side by A^{-1},

$$A^{-1}Ax^i = \lambda_i A^{-1}x^i$$

$$\Rightarrow A^{-1}x^i = \frac{1}{\lambda_i}x^i \qquad [\text{as } A^{-1}A = \mathbb{I}]$$

So, A^{-1} has the same eigenvectors x^i as does A, but the corresponding eigenvalues are $1/\lambda_i$

5.1. Eigenvalues and eigenvectors of Hermitian and anti-Hermitian matrices:-

- Prove that the eigenvalues of an Hermitian matrix are real

For any particular eigenvector x^i, we can write the equation-

$$Ax^i = \lambda_i x^i \quad \text{——— (I)}$$

Taking the Hermitian conjugate,

$$(x^i)^\dagger A^\dagger = \lambda_i^* (x^i)^\dagger \quad \text{——— (II)}$$

Since A is Hermitian $\rightarrow$ $A^\dagger = A$.

right-multiplying equation (II) with x^i—

$$(x^i)^\dagger A x^i = \lambda_i^* (x^i)^\dagger x^i$$

Now left-multiplying equation (I) by $(x^i)^\dagger$ gives

$$(x^i)^\dagger A x^i = \lambda_i (x^i)^\dagger x^i$$

So,

$$0 = (\lambda_i^* - \lambda_i)(x^i)^\dagger x^i$$

But, $(x^i)^\dagger x^i$ is the modulus squared of the non-zero vector x^i and is thus non-zero, So, $\lambda_i^* = \lambda_i$ i.e. λ_i must be "real."

Now, for anti-Hermitian matrix, we know that $A^\dagger = -A$.
So, in a similar manner, we can obtain:

$$-(x^i)^\dagger A x^i = \lambda_i^* (x^i)^\dagger x^i$$

and

$$(x^i)^\dagger A x^i = \lambda_i (x^i)^\dagger x^i$$

So,

$$0 = (\lambda_i^* + \lambda_i)(x^i)^\dagger x^i$$

$$\Rightarrow \boxed{\lambda_i^* = -\lambda_i} \rightarrow \lambda_i \text{ must be "pure imaginary or zero".}$$

- Prove that the eigenvectors corresponding to different eigenvalues of an Hermitian matrix are orthogonal.

Consider two unequal eigenvalues λ_i and λ_j and their corresponding eigenvectors satisfying —

$$Ax^i = \lambda_i x^i \quad \text{——— (I)}$$

$$Ax^j = \lambda_j x^j \quad \text{——— (II)}$$

Taking Hermitian conjugate of eq^n (I), we get $(x^i)^\dagger A^\dagger = \lambda_i^*(x^i)^\dagger$, ~~and~~ right-multiplying this by x^j we get —

$$(x^i)^\dagger A^\dagger x^j = \lambda_i^* (x^i)^\dagger x^j$$

$$\Rightarrow (x^i)^\dagger A x^j = \lambda_i^* (x^i)^\dagger x^j$$

Similarly, left-multiplying eq^n (ii) by $(x^i)^\dagger$, we get -

$$(x^i)^\dagger A x^j = \lambda_j (x^i)^\dagger x^j$$

So,

$$0 = (\lambda_i^* - \lambda_j)(x^i)^\dagger x^j$$
$$= (\lambda_i - \lambda_j)(x^i)^\dagger x^j$$ [As eigenvalues of Hermitian matrix are real, i.e. $\lambda_i^* = \lambda_i$]

As, $\lambda_i \neq \lambda_j$, so. $(x^i)^\dagger x^j = 0$, i.e. the eigenvectors x^i and x^j are orthogonal.

[The importance of these two results will be apparent in Quantum Mechanics, where the eigenvalues of any operators correspond to measured values of observable quantities (e.g. energy, angular momentum, parity etc.) and these clearly must be real. That's why in Quantum mechanics, people use Hermitian operators to ~~guaran~~ guarantee physically meaningful results.]

5.2. Eigenvalues and eigenvectors of a unitary matrix: —

$\Rightarrow$ Unitary matrix satisfies $A^\dagger = A^{-1}$

Now, $A x^i = \lambda_i x^i$ —— (i)

taking the Hermitian conjugate -

$$(x^i)^\dagger A^\dagger = \lambda_i^* (x^i)^\dagger$$

$$\Rightarrow (x^i)^\dagger A^{-1} = \lambda_i^* (x^i)^\dagger \text{ —— (ii)}$$

Now right-multiplying eq^n (ii) by eq^n (i), we get —

$$(x^i)^\dagger A^{-1} A x^i = \lambda_i^* (x^i)^\dagger \lambda_i x^i$$

$$\Rightarrow (x^i)^\dagger x^i = \lambda_i^* \lambda_i (x^i)^\dagger x^i$$

$$\Rightarrow \lambda_i^* \lambda_i = 1$$

$$\Rightarrow |\lambda_i|^2 = 1$$ $\longrightarrow$ Thus, the eigenvalues of a unitary matrix have unit modulus.

$\Rightarrow$ Now consider two different eigenvalues λ_i and λ_j :

$$A x^i = \lambda_i x^i \Rightarrow (x^i)^\dagger A^\dagger = \lambda_i^* (x^i)^\dagger \text{ (taking Hermitian conjugate) —— (i)}$$

$$A x^j = \lambda_j x^j \text{ —— (ii)}$$

Now, right-multiplying eq^n (i) by eq^n (ii) gives —

$$(x^i)^\dagger A^\dagger A x^j = \lambda_i^* \lambda_j (x^i)^\dagger x^j$$

$$\Rightarrow (1 - \lambda_i^* \lambda_j)(x^i)^\dagger x^j = 0 \quad [\text{using } A^\dagger A = I]$$

Since, $\lambda_i \neq \lambda_j \Rightarrow \lambda_i^* \lambda_j \neq 1$, so, $(x^i)^\dagger x^j = 0 \Rightarrow$ Eigenvectors corresponding to different eigenvalues are mutually orthogonal.

5.3. Determination of eigenvalues and eigenvectors:

How to determine eigenvalues and corresponding eigenvectors for an $N \times N$ matrix A?

$$Ax - \lambda I x = 0 \qquad [\, I \text{ is the Unit matrix of order } N \,]$$

$$\Rightarrow (A - \lambda I)x = 0$$

$\Rightarrow$ It can be shown that, this equation only has a non-trivial solution for x if $\det(A - \lambda I) = 0$, i.e.

$$\boxed{|A - \lambda I| = 0}$$

$\longrightarrow$ This equation is known as the "characteristic equation" for A.

Two important relations of eigenvalues with the matrix:

(I) $\sum_{i=1}^{N} \lambda_i \equiv$ sum of eigenvalues $= \mathrm{Tr}\, A$

(II) $\prod_{i=1}^{N} \lambda_i \equiv$ product of eigenvalues $= |A|$

$\longrightarrow$ These can be used to check whether the eigenvalues are computed correctly.

Ex: Find the eigenvalues and normalised eigenvectors of the real symmetric matrix

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & -3 \\ 3 & -3 & -3 \end{pmatrix}$$

The characteristic equation —

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 1-\lambda & -3 \\ 3 & -3 & -3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[(1-\lambda)(-3-\lambda) - 9\right] + 1\left[-9 - 1(-3-\lambda)\right] + 3\left[-3 - 3(1-\lambda)\right] = 0$$

$$\Rightarrow (1-\lambda)\left[-3 - \lambda + 3\lambda + \lambda^2 - 9\right] + \left[\lambda - 6\right] + 3\left[-6 + 3\lambda\right] = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2 + 2\lambda - 12) + (\lambda - 6) + 9\lambda - 18 = 0$$

$\Rightarrow \lambda^3 + \lambda^2 - 24\lambda + 36 = 0$

Now, to solve this cubic eq^n, we first look for integer valued solutions. Such solutions divide the constant term 36. So, the list of possible integer solutions is :

$$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36.$$

If we try $\lambda = 2 \Rightarrow 8 + 4 - 48 + 36 = 0$

so, $\lambda = 2$ is one of the solutions.

So, let's factor out $(\lambda - 2)$:

$$\lambda^3 + \lambda^2 - 24\lambda + 36 = 0$$

$$\Rightarrow \lambda^3 - 2\lambda^2 + 3\lambda^2 - 24\lambda + 36 = 0$$

$$\Rightarrow \lambda^2(\lambda - 2) + 3(\lambda^2 - 8\lambda + 12) = 0$$

$$\Rightarrow \lambda^2(\lambda - 2) + 3\lambda(\lambda - 2) - 3.6(\lambda - 2) = 0$$

$$\Rightarrow (\lambda - 2)(\lambda^2 + 3\lambda - 18) = 0$$

Similarly, we can solve $\lambda^2 + 3\lambda - 18 = 0$, list of possible inter sol^{n}s is : $\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18$.

We find : $(\lambda - 2)(\lambda - 3)(\lambda + 6) = 0$.

OR, to solve $\lambda^2 + 3\lambda - 18 = 0$, we can use the sol^n of quadratic eq^n like : $ax^2 + bx + c = 0$

sol^n : $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

here, $a = 1$, $b = 3$, $c = -18$

So, $\lambda = \dfrac{-3 \pm \sqrt{9 + 72}}{2} = \dfrac{-3 \pm 9}{2} = -6$ or 3

$\therefore \lambda = 3$ or $\lambda = -6$.

So, the three eigenvalues are $\Rightarrow \boxed{\lambda_1 = 2,\ \lambda_2 = 3 \text{ and } \lambda_3 = -6.}$

Note that, $\lambda_1 + \lambda_2 + \lambda_3 = \text{Tr } A \ (= -1)$

and $\lambda_1 \lambda_2 \lambda_3 = |A| \ (= -36)$

Now calculate the eigenvectors.

- For $\lambda_1 = 2$, let's say the eigenvector is $x' = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

this eigenvector x' must satisfy the eigenvector-eqⁿ:

$$Ax' = 2x', \text{ or equivalently} -$$

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & -3 \\ 3 & -3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{pmatrix}$$

i.e.

$$x_1 + x_2 + 3x_3 = 2x_1 \quad \text{—— (i)}$$

$$x_1 + x_2 - 3x_3 = 2x_2 \quad \text{—— (ii)}$$

$$3x_1 - 3x_2 - 3x_3 = 2x_3 \quad \text{—— (iii)}$$

(i) − (ii) $\longrightarrow 3x_3 = \cancel{0}\,(x_1 - x_2)$. So, from eqⁿ (iii), $x_3 = 0$ and $x_1 = x_2 = k$, where k is any non-zero number.

So, $$x' = \begin{pmatrix} k \\ k \\ 0 \end{pmatrix}$$

If we apply normalization condition, i.e. $(x')^T x' = 1$

$$\Rightarrow (k \;\; k \;\; 0) \begin{pmatrix} k \\ k \\ 0 \end{pmatrix} = 1$$

$$\Rightarrow k^2 + k^2 + 0 = 1$$

$$\Rightarrow 2k^2 = 1$$

$$\Rightarrow k = \frac{1}{\sqrt{2}}$$

So, $$x' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Similarly, we can find,

for $\lambda_2 = 3$, $$x^2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

and for $\lambda_3 = -6$, $$x^3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

As this matrix A is Hermitian, Check that, these ~~three~~ eigenvectors are orthogonal.

i.e. $(x^1)^T x^2 = (x^1)^T x^3 = (x^2)^T x^3 = 0$

Degenerate eigenvalues:

If two or more eigenvalues are same.

Ex: Construct an orthonormal set of eigenvectors for the matrix

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

First determine $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 3 \\ 0 & -2-\lambda & 0 \\ 3 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)^2(-2-\lambda) + 3.3.(2+\lambda) = 0$$

$$\Rightarrow (2+\lambda)\left[-(1-\lambda)^2 + 9\right] = 0.$$

$$\Rightarrow (2+\lambda)\left[3^2 - (1-\lambda)^2\right] = 0$$

$$\Rightarrow (2+\lambda)(3+1-\lambda)(3-1+\lambda) = 0$$

$$\Rightarrow (2+\lambda)^2(4-\lambda) = 0.$$

So, $\boxed{\lambda_1 = 4, \quad \lambda_2 = -2 = \lambda_3} \Longrightarrow$ degenerate!

for $\lambda_1 = 4$, say the eigenvector $X^1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

which satisfies $AX^1 = \lambda_1 X^1 = 4X^1$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{pmatrix}$$

i.e.

$$x_1 + 3x_3 = 4x_1 \quad \text{—— (i)}$$

$$-2x_2 = 4x_2 \quad \text{—— (ii)}$$

$$3x_1 + x_3 = 4x_3 \quad \text{—— (iii)}$$

for eq^n (ii), $x_2 = 0$

(i) $\times 3 -$ (iii) $\longrightarrow 9x_3 - x_3 = 12x_1 - 4x_3.$

$\Rightarrow x_1 = x_3 = k$ (where k is any non-zero number)

So, $x^1 = \begin{pmatrix} k \\ 0 \\ k \end{pmatrix}$

after normalizing $x^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Now, we have to find other two eigenvectors x^2 and x^3 which are orthogonal with x^1 and each other.

A general column vector that is orthogonal to x^1 is

$$x = \begin{pmatrix} a \\ b \\ -a \end{pmatrix} \quad [\text{check that, } (x^1)^T x = 0]$$

$$Ax = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ -a \end{pmatrix} = \begin{pmatrix} a-3a \\ -2b \\ 3a-a \end{pmatrix} = \begin{pmatrix} -2a \\ -2b \\ 2a \end{pmatrix}$$

$$= -2 \begin{pmatrix} a \\ b \\ -a \end{pmatrix} = -2x$$

thus, x is a eigenvector of A with eigenvalue -2.

However, there are infinite sets of eigenvector x all possessing the required property. Therefore, let us choose a simple form for x^2, suitably normalized, say,

$$x^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

The third eigenvector is then specified! x^3 has to be orthogonal to both x^1 and x^2, i.e. $(x^1)^T x^3 = 0 = (x^2)^T x^3$

let's say, $x^3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

So, $(x^1)^T x^3 = 0$

$\Rightarrow \frac{1}{\sqrt{2}} (1 \;\; 0 \;\; 1) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$\Rightarrow x_1 + x_3 = 0$

and, $(x^2)^T x^3 = 0$

$\Rightarrow (0 \;\; 1 \;\; 0) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$\Rightarrow x_2 = 0$.

Say, $x_1 = -x_3 = k$ (k is non-zero number).

So, $x^3 = \begin{pmatrix} k \\ 0 \\ -k \end{pmatrix}$

After normalizing, $x^3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

6) Change of basis and Similarity transformation :—

Similarity transformation is a linear transformation due to the change of basis, which is very useful in diagonalization of a matrix.

A square matrix A is similar to another square matrix A′ if there exists a non-singular matrix S such that –

$$\boxed{A' = S^{-1} A S} \longrightarrow$$ A′ represents the same linear operator A in a new basis.

where the new basis $\hat{e}_j'$ is related to the old basis $\hat{e}_i$ by –

$$\hat{e}_j' = \sum_i S_{ij}\, \hat{e}_i \,.$$

Thus, we expect the properties of A will also be shared by the new matrix A′.

(i) If $A = I$, then $A' = I$:

$$\Rightarrow \quad A' = S^{-1} I S = S^{-1} S = I$$

(ii) The value of the determinant is unchanged;

$$\Rightarrow |A'| = |S^{-1} A S| = |S^{-1}|\,|A|\,|S|$$
$$= |A|\,|S^{-1}|\,|S| = |A|\,|S^{-1}S| = |A| .$$

(iii) The "characteristic determinant" and hence the eigenvalues of A′ are the same as those of A :

$$\Rightarrow |A' - \lambda I| = |S^{-1}AS - \lambda I| = |S^{-1}AS - \lambda S^{-1} I S| \quad [\text{as } S^{-1}IS = S^{-1}S = I]$$
$$= |S^{-1}(A - \lambda I)S|$$
$$= |S^{-1}|\,|S|\,|A - \lambda I| = |A - \lambda I|$$

(iv) The trace is unchanged.

$$Tr(A') = Tr(A) .$$

- **Unitary transformation:**

A special class of similarity transformation for which S is a unitary matrix :

i.e. $$\boxed{A' = S^{-1} A S = S^{\dagger} A S}$$

⇒ If the original basis $\hat{e}_i$ is orthonormal and the transformation matrix S is unitary, then the new basis $\hat{e}_i'$ is also orthonormal.

For Unitary transformations —

(i) If A is Hermitian (anti-Hermitian) then A' is also Hermitian (anti-Hermitian), i.e.

if $A^\dagger = \pm A$ then

$$(A')^\dagger = (S^\dagger A S)^\dagger = S^\dagger A^\dagger S = \pm S^\dagger A S = \pm A'$$

(ii) If A is unitary (i.e. $A^\dagger = A^{-1}$), then A' is also unitary.

$$\begin{aligned}(A')^\dagger A' &= (S^\dagger A S)^\dagger (S^\dagger A S) \\ &= S^\dagger A^\dagger S S^\dagger A S \\ &= S^\dagger A^\dagger A S \quad [\text{as } S^\dagger = S^{-1}] \\ &= S^\dagger I S = I.\end{aligned}$$

7 Diagonalisation of matrices :—

Say, in a new basis, a matrix A' is related to a matrix A of its old basis by a similarity transformation —

$$A' = S^{-1} A S.$$

Given the matrix A, if we construct the matrix S that has the eigenvectors of A as its columns, then the new matrix $A' = S^{-1} A S$ will be diagonal and has the eigenvalues of A as its diagonal elements. [S must be non-singular].

If, $S = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ x^1 & x^2 & \cdots & x^N \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$ where $x^1, x^2, \cdots x^N$ are eigenvectors of A with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_N$.

then $A' = S^{-1} A S = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{pmatrix}$

→ This procedure is only valid for ~~linear~~ a matrix A having linearly independent eigenvectors.

If the eigenvectors are linearly independent and the matrix is normal (i.e. Hermitian, anti-Hermitian or Unitary, (or symmetric, anti-symmetric, orthogonal)), then we ~~have to~~ can use a unitary transformation.

~~For real symmetric matrix, we use a similarity transformation to diagonalize it.~~ unitary

Ex: Diagonalize the matrix

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

→ The matrix A is symmetric, so we can use $A' = S^{\dagger} A S$.
We're already found the eigenvectors of A.
So, we can write -

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$\left[A_S, \ \lambda \begin{pmatrix} A_{11} & A_{12} & \cdots \\ A_{21} & A_{22} & \cdots \\ \vdots & & \end{pmatrix} = \begin{pmatrix} \lambda A_{11} & \lambda A_{12} & \cdots \\ \lambda A_{21} & \lambda A_{22} & \cdots \\ \vdots & & \end{pmatrix} \right]$$

$$A' = S^{\dagger} A S = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 4 & 0 & -2 \\ 0 & -2\sqrt{2} & 0 \\ 4 & 0 & 2 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 8 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

So, A' is diagonal and its diagonal elements are the eigenvalues of A.

Ex: Calculate the eigenvalues & eigenvectors of

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

and find a matrix S, such that $S^{\dagger} A S$ is diagonal.

→ $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(2-\lambda)(1-\lambda) - (2-\lambda) = 0$$

$$\Rightarrow (2-\lambda)(1 - 2\lambda + \lambda^2 - 1) = 0$$

$$\Rightarrow (2-\lambda)(\lambda^2 - 2\lambda) = 0$$

$$\Rightarrow \lambda(\lambda-2)^2 = 0$$

$\lambda_1 = 0, \ \lambda_2 = \lambda_3 = 2$ (degenerate — 2-fold degeneracy)

For $\lambda_1 = 0$, say, $x_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$Ax_1 = \lambda_1 x_1$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} a \quad + c &= 0 \\ 2b \quad &= 0 \\ a \quad + c &= 0 \end{aligned}\right\} \quad \begin{aligned} b &= 0 \\ a = -c &= k \text{ (say)} \end{aligned}$$

So, $x_1 = \begin{pmatrix} k \\ 0 \\ -k \end{pmatrix}$, normalized $\boxed{x_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}$

For $\lambda_2 = 2$, say $x_2 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$A x_2 = \lambda_2 x_2$$

$$\Rightarrow \begin{pmatrix} 1-2 & 0 & 1 \\ 0 & 2-2 & 0 \\ 1 & 0 & 1-2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} -a \quad + c &= 0 \\ a \quad - c &= 0 \end{aligned}\right\} \quad a = c, \ b \text{ is unknown}$$

infinite no. of solutions are possible.

For convinience, take $b = 0$ and $a = c = 1$

So, the normalized $\boxed{x_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}$

now to determine x_3, we'll assume that x_3 is orthogonal to both x_1 and x_2 (even if the matrix A is Hermitian, but the eigenvectors corresponding to "different" eigenvalues are orthogonal, but here eigenvectors with the same eigenvalues are taken as orthogonal. We can still do that, as we can transform the basis of A in such a way that in the new basis x_2 and x_1 are orthogonal to x_3)

let's say, $x_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$(x_1)^T x_3 = 0 \Rightarrow (1 \ \ 0 \ -1)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$\Rightarrow a = c$$

$$(x_2)^T x_3 = 0 \Rightarrow (1 \ \ 0 \ \ 1)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$\Rightarrow a = -c$$

So, $a = 0 = c$

Take $b = 1$ (to ensure x_3 is automatically normalized)

We cannot take $b = 0$, then $x_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. trivial sol^n.

The normalized $\boxed{x_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$

So, the matrix S will be:

$$S = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ -1 & 1 & 0 \end{pmatrix}$$

We can apply the "Unitary transformation":

$$A' = S^{+} A S$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ -1 & 1 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2\sqrt{2} \\ 0 & 2 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

A' is diagonal with eigenvalues are diagonal elements

8 Cayley - Hamilton Theorem :

Every square matrix satisfies its own characteristic equation.

If A is an $n \times n$ matrix and its characteristic polynomial $p(\lambda)$ is written as :

$$p(\lambda) = \det(A - \lambda I) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$
$$(I \equiv n \times n \text{ identity matrix})$$

Then Cayley - Hamiltonian theorem says that the matrix A also satisfies its own characteristic polynomial, i.e.

$$p(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I$$

This polynomial results in a "zero matrix"

i.e. $p(A) = \mathbf{0}$, where $\mathbf{0}$ is $n \times n$ zero matrix.

- If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the characteristic polynomial of A is given by

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}$$
$$= \lambda^2 - (a+d)\lambda + (ad - bc)$$

that is,

$$p(\lambda) = \lambda^2 - tr(A)\,\lambda + \det(A)$$
$$tr(A) = \text{trace of } A = a + d$$
$$\det(A) = ad - bc$$

From the Cayley - Hamilton theorem we have

$$\boxed{p(A) = A^2 - tr(A) \cdot A + \det(A) \cdot I = \mathbf{0}}$$

for any 2×2 matrix A.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

- Example

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}, \quad p(\lambda) = \begin{vmatrix} 2-\lambda & 3 \\ 0 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) = \lambda^2 - 6\lambda + 8$$

So, Cayley - Hamilton theorem says that, $A^2 - 6A + 8I = 0$

9 Trick to find inverse of a matrix easily:

I First method:

Let, $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 10 \end{pmatrix}$, find A^{-1}

We'll first write the matrix A in an "augmented matrix" form with an identity (I) matrix of the same rank, like –

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ 3 & 6 & 10 & 0 & 0 & 1 \end{array}\right)$$

(*) Offcource $|A| \neq 0$. First check that!

we'll try to ~~create~~ make this part as the identity matrix by some basic row/column operations.

we'll make the same operations in this part as well. This part will become A^{-1}

$$\Downarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & a & b & c \\ 0 & 1 & 0 & d & e & f \\ 0 & 0 & 1 & g & h & i \end{array}\right)$$

We'll set $A^{-1} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$.

Let's do it.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ 3 & 6 & 10 & 0 & 0 & 1 \end{array}\right)$$

$R \equiv$ row
$C \equiv$ column.

try to make these two elements zero first.

① $R_2 - 2R_1 \leftrightarrow R_2$
② $R_3 - 3R_1 \leftrightarrow R_3$

the matrix will be unchanged following the property of matrix row-colum operations.

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right)$$

Next aim to make these two elements zero.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right)$$

(3) $R_1 - 2R_2 \leftrightarrow R_1$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 5 & -2 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right)$$

Next, (4) $R_1 - 3R_3 \leftrightarrow R_1$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 14 & -2 & -3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right)$$

Identity matrix.

So, $A^{-1} = \begin{pmatrix} 14 & -2 & -3 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$ Check ~~of~~ $A^{-1}A = \mathbb{1}$

II) 2nd Method:

Let $A = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$, find A^{-1}

For this method, we'll use the formula for inverse matrix.

$$A^{-1} = \frac{1}{|A|} \text{Adj}(A)$$

$$|A| = 1(3-0) + 2(1) - 2(+2)$$

$$= 3 + 2 - 4 = 1 \neq 0, \text{ so, } A^{-1} \text{ exists.}$$

Now we'll use a trick to calculate the adjoint of A.

(I) First write the entire matrix as it is

(II) Then repeat C_1 & C_2 as the 4th and 5th column.

(III) Then copy the R_1 & R_2 and write them as 4th and 5th row.

$$\begin{array}{c|ccccc} & C_1 & C_2 & & & \\ \hline R_1 & 1 & 2 & -2 & 1 & 2 \\ R_2 & -1 & 3 & 0 & -1 & 3 \\ & 0 & -2 & 1 & 0 & -2 \\ & 1 & 2 & -2 & 1 & 2 \\ & -1 & 3 & 0 & -1 & 3 \end{array}$$

(IV) Then cancel out the first row and first column.

$$\begin{array}{c|cccc} 1 & 2 & -2 & 1 & 2 \\ \hline -1 & 3 & 0 & -1 & 3 \\ 0 & -2 & 1 & 0 & -2 \\ 1 & 2 & -2 & 1 & 2 \\ -1 & 3 & 0 & -1 & 3 \end{array}$$

(V) Now calculate the adj (A) (like determinant) from the remaining matrix by cross multiplication. Take a box matrix in column wise and write down the result as a single element ~~of that row or column~~ in the row.

(similar to taking transpose)

$$\begin{array}{cccc} 3 & 0 & -1 & 3 \\ -2 & 1 & 0 & -2 \\ 2 & -2 & 1 & 2 \\ 3 & 0 & -1 & 3 \end{array}$$

For the boxes in 1st column, we'll get the 1st row.

$$\begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix} = \text{Adj}(A).$$

For the 2nd row, take boxes like this from the column.

$$\begin{array}{cccc} 3 & 0 & -1 & 3 \\ -2 & 1 & 0 & -2 \\ 2 & -2 & 1 & 2 \\ 3 & 0 & -1 & 3 \end{array}$$

and similarly for the 3rd row.

So, $A^{-1} = \frac{1}{1}\begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}$

10 Matrix method for solving linear systems of equations:

Suppose there are N no. of simultaneous linear equations in N unknowns ~~of the forms~~ $x_1, x_2, \ldots, x_N$ of the form:

$$\begin{aligned}
A_{11} x_1 + A_{12} x_2 + \cdots + A_{1N} x_N &= b_1 \\
A_{21} x_1 + A_{22} x_2 + \cdots + A_{2N} x_N &= b_2 \\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\quad \vdots \\
A_{N1} x_1 + A_{N2} x_2 + \cdots + A_{NN} x_N &= b_N
\end{aligned}$$

where, A_{ij} and b_i are known.

(*) If "all" the b_i are zero, then the system of equations is called "homogeneous", otherwise it is "inhomogeneous".

This set of equations may be expressed as a single matrix equation

$$\boxed{Ax = b}$$

or,
$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix}$$

Depending on the given values of A_{ij} and b_i, this set of equations may have either a unique solution, no solution or infinitely many solutions.

Matrix notation on solving system of equations:

Consider the simplest algebric equation : $ax = b$, where a, b are constants (known) and x is unknown. We easily solve this for x as $x = a^{-1} b$ (provided $a \neq 0$)

We'll apply the same notation for system of equations but now with help of matrix.

For example, consider the following system of two equations with two unknowns (x, y):

$$6x + 5y = 2$$
$$x + 2y = 3.$$

In matrix form, we can write this as:

$$\begin{pmatrix} 6 & 5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Solution $\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 1 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Let, $A = \begin{pmatrix} 6 & 5 \\ 1 & 2 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

So, $\boxed{AX = b}$

$\Rightarrow \boxed{X = A^{-1} b} \leftarrow$ solution.

Similarly for differential equations like:

$$x' = 6x + 5y$$
$$y' = x + 2y$$

we can write in matrix form as:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow X' = AX \leftarrow$ this is a linear, homogeneous, system of ODEs.

How to solve coupled linear ordinary homogeneous DEs.?

- **Matrix method:**

Consider the system of ODE's (homogeneous):

$$x' = 6x + 5y \qquad y' = x + 2y. \qquad \Big| \text{ here } ' \equiv \frac{d}{dt}$$

In order to solve this, write it in matrix form:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow X' = AX$

As this is a homogeneous ODE, we'll try the solution as

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{\lambda t} V$$

where $\underbrace{\lambda = (\lambda_1, \lambda_2)}_{\text{eigenvalues}}$, $V = \underbrace{\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}}_{\text{eigenvectors}}$

The matrix method tells us that we need to find the eigenvalues of matrix A, then we will be able to ~~diagonalize it and find~~ calculate the eigenvectors so that we can write the solution as a linear combination of the eigenvectors that we found. (For reason behind this, see the next example)

So, let's calculate the eigenvalues of A first from the characteristic eq^n.

$$\begin{vmatrix} 6-\lambda & 5 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (6-\lambda)(2-\lambda) - 5 = 0$$

$$\Rightarrow \quad \lambda^2 - 8\lambda + 7 = 0$$

Sol^n: $\lambda_1 = 1$ & $\lambda_2 = 7$.

Now calculate the eigenvectors associated with these eigenvalues.

for $\lambda_1 = 1$
eigenvector $\vec{V_1}$

$$\begin{pmatrix} 6-1 & 5 \\ 1 & 2-1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 5 & 5 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{l} 5a_1 + 5a_2 = 0 \\ a_1 + a_2 = 0 \end{array}\right] \text{- same equation.}$$

so, $a_1 = -a_2$

the simplest choice would be, $a_1 = 1$ so $a_2 = -1$

$$\vec{V_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \longleftarrow \text{you can normalize it if you want.}$$

Now, for $\lambda_2 = 7$
eigenvector $\vec{V_2}$

$$\begin{pmatrix} 6-7 & 5 \\ 1 & 2-7 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1 & 5 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{l} -a_1 + 5a_2 = 0 \\ a_1 - 5a_2 = 0 \end{array}\right] \text{— same equation.}$$

so, $a_1 = 5a_2$, choose $a_2 = 1$, so $a_1 = 5$

$$\vec{V_2} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \longleftarrow \text{you can again normalize this if you want.}$$

So, the solution of the original equations will be a linear combination of these two eigenvectors in the form of $e^{\lambda t} v$

<u>Solution;</u>

$$X = \begin{pmatrix} x \\ y \end{pmatrix} = c_1 X_1 + c_2 X_2$$

where, $X_1 = e^{\lambda_1 t} v_1 = e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$X_2 = e^{\lambda_2 t} v_2 = e^{7t} \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

So, the general solution is :

$$X = \begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{7t} \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

c_1, c_2 are constants which can be determined if boundary ~~cond~~ or initial conditions are provided.

Note that, ~~the normali~~ if we normalize the eigenvectors, the normalization constants would be absorbed in c_1 and c_2. That's why we didn't normalize them.

<u>Ex1:</u> Solve

$$x' = 3x + 2y$$
$$y' = x + 2y$$

<u>Sol^n:</u> Try $\begin{pmatrix} x \\ y \end{pmatrix} = e^{\lambda t} v$

substitution gives $\lambda e^{\lambda t} v = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} e^{\lambda t} v$

$$\Rightarrow \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} v = \lambda v \Rightarrow \underbrace{Av = \lambda v}_{\text{eigenvalue equation.}}$$

That's why we need to determine eigenvalues and eigenvectors of A.

eigenvalues $\lambda = 4, 1$.

for $\lambda_1 = 4$, $v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

for $\lambda_2 = 1$, $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\rightarrow$ Check these!

$X_1 = e^{4t} v_1$ and $X_2 = e^{t} v_2$

So, the general solution is,

$$X = \begin{pmatrix} x \\ y \end{pmatrix} = c_1 X_1 + c_2 X_2 = c_1 e^{4t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$